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TRAINING IN HIGH-SCHOOL MATHEMATICS ESSENTIAL FOR SUCCESS IN CERTAIN COLLEGE SUBJECTS

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CONTENTS

CHAPTER		PAGE
	INTRODUCTION	vii
I.	SUMMARY OF OTHER INVESTIGATIONS PERTAINING TO THIS SUBJECT	1
II.	MATHEMATICAL VOCABULARY	7
III.	SYMBOLISM	15
IV.	FACTS, CONCEPTS, AND SKILLS	31
V.	GENERAL PROCESSES AND METHODS OF PROCEDURE	57
VI.	TEST ON FORMULAS	77
VII.	SUMMARY AND CONCLUSIONS	89
	APPENDIX	97
	BIBLIOGRAPHY	101

INTRODUCTION

From numerous reports of work done in experimental courses in high-school mathematics, it is evident that many progressive teachers have become dissatisfied with the old formal courses and are endeavoring to develop something better. This apparent dissatisfaction with both subject matter and method is the most encouraging feature of the mathematics situation to-day.

The readiness with which failures in college and high-school subjects are attributed to inefficient teaching in the preceding subjects has become proverbial. Some of the reports to which reference is made later in this discussion indicate the attitude of college teachers toward high-school mathematics as one of these preceding subjects. The fact that failures in college subjects are so frequently attributed to unsatisfactory work in high-school mathematics should be accepted as both a compliment and a challenge. It is gratifying to know that many college professors feel that training in mathematics is essential for success in their subjects. On the other hand, there is a challenge to the teacher of high-school mathematics to determine just wherein the presentation of the subject may be improved in order to reduce to a minimum the difficulties met in college work.

It is with the hope of partially answering the questions raised by this challenge that the present study has been undertaken. The purpose of the study is to determine what training in high-school mathematics is essential for success in certain college subjects that do not presuppose a knowledge of college mathematics. *Training in high-school mathematics* is interpreted to include the various outcomes from the study of algebra, geometry, and trigonometry, such as facts, concepts, skills, general processes, and methods of procedure. It is true that some of the high schools do not teach trigonometry, and it is also true that many of them are successfully introducing the elementary principles of analytic geometry and the calculus. Algebra, geometry, and trigonometry have been chosen, not because they are universally taught, but because a knowledge of these three subjects is generally assumed by those

who teach the college subjects that presuppose a good background of high-school mathematics. Courses in college mathematics are not included in the college subjects selected for consideration in this study. Justification for this omission is found in the following quotation from the Report of the National Committee on Mathematical Requirements [1] (hereinafter referred to as the National Committee Report):

> Mathematical ability appears to be a sufficient but not a necessary condition for general intelligence. For this, as well as for other reasons, it would appear that college entrance requirements in mathematics should be formulated primarily on the basis of the special knowledge and training required for the successful study of courses which the student will take in college. . . . In the selection of material of instruction for high-school courses in mathematics, its value as preparation for college courses in mathematics need not be specifically considered. Not all college students study mathematics; it is therefore reasonable to expect college departments in this subject to adjust themselves to the previous preparation of their students. Nearly all college students do, however, study one or more of the physical sciences (astronomy, physics, chemistry) and one or more of the social sciences (history, economics, political science, sociology). Entrance requirements must therefore insure adequate mathematical preparation in these subjects. Moreover, it may be assumed that adequate preparation for these two groups of subjects will be sufficient for all other subjects for which the secondary schools may be expected to furnish the mathematical prerequisites.

After a summary (Chapter I) of the work done by other investigators on questions pertaining to the relationship between high-school mathematics and other subjects, the present investigation makes use of Stewart's [2] *Physics* as the chief source of data. This text has been in use a sufficient length of time to be fairly well known, and yet is not old enough to be considered out of date. It was used as the basis of study in the following analyses:

1. The mathematical vocabulary used has been determined by a careful word count for the entire text.

2. The symbolism employed in the text has been examined, tabulated, and compared with that of other texts in physics and chemistry.

3. By means of an intensive study of the entire text, including the complete solution of the five hundred seventy-two problems

[1] *The Reorganization of Mathematics in Secondary Education*, A Report of the National Committee on Mathematical Requirements, pp. 43–44, Dartmouth Press, 1923.

[2] Stewart, Oscar M., *Physics, A Textbook for Colleges*. Ginn and Company, 1924.

presented in the text, the facts, concepts, skills, general processes, and methods of procedure which are used have been analyzed and tabulated.

Tabulations of the results are presented and discussed in Chapters II, III, IV, and V.

Other investigations related to the present study have dealt primarily with the actual content of the high-school courses in mathematics. The elimination of the obsolete material in algebra courses, and the introduction of new material is a much-needed reform that has been brought about by a variety of influences.[3] The term "training" in the preceding quotation from the National Committee Report, and in the title of this dissertation implies, however, something more than content. "Methods of attack" more nearly expresses the idea. A certain college professor indicated that he was fairly well satisfied with the knowledge of mathematics possessed by his students in freshman physics but lamented the fact that they did not know how to attack a problem. As he expressed it, "They have not been trained in *the ways of thinking* which they need in physics." The determination of the *facts, concepts,* and *skills* from mathematics which are used in college subjects is a comparatively easy matter. The task of ascertaining the general processes and methods of procedure is more difficult, but none the less important. The writer attempted to discover what this physics professor termed *the ways of thinking,* first, by studying the text and solving the problems, and then by making a sort of introspective analysis in an endeavor to identify the processes and procedures that are involved.

Finally, from the list of algebraic manipulations actually occurring in this text and in the solution of the problems, eighteen problems representative of the various skills involved were selected and arranged to form a test. This test does not purport to be a survey of material that should be covered in high school, but involves merely the types of problems that actually occur in college physics. The test was given to eight hundred fifty-nine high-school pupils. From an analysis of the results in this test together with the other tabulated findings, the summary and final conclusions have been established.

[3] For a complete discussion of the significant changes and trends in the teaching of mathematics in the United States, consult the article written by W. D. Reeve in the *Fourth Yearbook of the National Council of Teachers of Mathematics,* pp. 131–86. Bureau of Publications, Teachers College, Columbia University, 1929.

TRAINING IN HIGH-SCHOOL MATHEMATICS ESSENTIAL FOR SUCCESS IN CERTAIN COLLEGE SUBJECTS

CHAPTER I

SUMMARY OF OTHER INVESTIGATIONS PERTAINING TO THIS SUBJECT

An important attack on the problem of ascertaining what training in high-school mathematics is essential for the studying of college subjects has been made by the National Committee on Mathematical Requirements.[1] In order to determine the relative importance of various portions of mathematics as preparation for college subjects, the Committee sent a selected list of forty-one topics in mathematics to a number of college teachers prominent in their respective fields, with the request that each one evaluate the topics as *essential, of considerable value, of small value,* or *of little or no value* as preparation for the elementary courses in his subject. The National Committee's tabulation of responses is reproduced in Table 2 of the Appendix. Reference will be made to this tabulation in connection with another part of this study. Thorndike[2] has made a very illuminating analysis of the findings of the National Committee. He calls attention to the great variation with respect to the rating of practically every topic, and also points out that: "Such ratings are the product partly of impartial observations, and partly of certain loyalties and convictions."

As a result of the announcement of the preliminary report of the National Committee, an elaborate questionnaire study was launched by the Mathematics Section of the New York Society for the Experimental Study of Education in January, 1921. Unfortunately, the complete report of the results of this study has never been published. A brief account[3] appeared in 1924. This

[1] National Committee Report, pp. 44–46.

[2] Thorndike, E. L. and others, *The Psychology of Algebra,* pp. 48–58. The Macmillan Company, 1923.

[3] Koch, E. H. and Schlauch, W. S., "Topics in High School Mathematics," in *Contributions to Education,* of the New York Society for the Experimental Study of Education, Volume One, pp. 351–59. World Book Company, 1924.

questionnaire was sent to: "Five hundred college professors in the United States who represented departments of astronomy, mathematics, physics, chemistry, mechanical engineering, civil engineering, zoölogy, biology, economics, accounting, and English." Through the kindness of the authors, access to the original complete manuscript was made possible. Approximately one hundred twenty-five replies were received, of which seventy-seven were sufficiently complete to be used. Thirty-seven different institutions were represented. Each professor was asked to number the items in order of importance, and also to place an *e* opposite each item considered essential for his subject and an *x* opposite those items which he would like to see omitted from the secondary school syllabus. Through rather complicated statistical techniques fully explained in the complete report, the authors combined the numerical and literal evaluations in order to divide the topics into six groups according to their importance. These groups are shown in Table 1 of the Appendix. The items in the first four groups are considered essential. In addition to this part of the questionnaire, which very closely paralleled the work of the National Committee, there were three other parts. One concerned typical mathematical errors. A list of ten types of errors was submitted with the request that they be numbered in order of frequency of occurrence. As this arrangement was based entirely upon opinion, its significance may be questioned. Another part, dealing with the relationship between mathematical preparation and failure in college is interesting. Twenty-one of the twenty-eight professors who reported attributed from five per cent to ninety-five per cent of the failures of their students to inadequate preparation in high-school mathematics. The remaining part is entitled *Tests,* and is an attempt to ascertain what is being done by college professors to determine the fitness of students to carry their courses. It does not reveal that much was being done at the time the study was conducted.

Thorndike [4] sent a questionnaire consisting of two parts to several college teachers of science in order to obtain their judgments as to the relative importance of certain portions of algebra. One part consisted of fifty-six specific computational tasks which these teachers were asked to rate, using the same terms as those used in the investigation by the National Committee. Of these

[4] Thorndike and others, *op. cit.*, pp. 59–69.

fifty-six tasks, thirty-seven were equations of the sort ordinarily seen in algebra texts. Three had values of x and y given for the equation $y = mx + b$, and questions were asked about m and b. One dealt with the discriminant of a quadratic equation. The remaining fifteen tasks involved radicals and exponents, including standard form numbers. The other part of the questionnaire consisted of nineteen verbal problems which were rated by ten college teachers. The following comment by Thorndike is significant:

> There was a tendency to confuse the ability to do one of these tasks with the ability to learn to do it or with the general ability of which the ability to do one of these tasks is a symptom.

The general conclusion is that physics students should have a thorough mastery of algebraic computation, but that students of the biological and social sciences, with the exception of economics and statistics, make little use of algebraic techniques. The discovery of the values of the so-called *constants* in simultaneous equations when the values of the *variables* are given and the use of standard form numbers receive a ranking that would seem to justify more emphasis upon them than has been given in high-school algebra courses. The five teachers of chemistry and physics consider verbal problems of great importance, while the teachers of psychology, economics, and sociology deem them of little value.

In order to discover the uses of mathematics in freshman college chemistry, Williams [5] made a careful analysis of the text written by W. A. Noyes, entitled *A Textbook in Chemistry*. The greater portion of his study deals with arithmetical forms of computation and notation which are not pertinent to the present study. The mathematical vocabulary determined by a word count consists largely of words occurring in arithmetic. Table I shows the items from Williams' study which are of chief interest to us.

The six cases of clearing of fractions have integral numerical denominators. All of the proportions are of the following type:

$$60.3 : 3 :: 48.6 : ?$$

Nearly one thousand symbols and formulas are used, and about five hundred equations appear. The chemical formulas and equations are applications of a symbolic language, but not like the algebraic formulas and equations.

[5] Williams, L. W., "The Mathematics Needed in Freshman Chemistry," *School Science and Mathematics*, Vol. XXI, pp. 654–65, October, 1921.

A study similar to that by Williams was made by Mrs. T. T. Calloway on a textbook entitled *Clothing for Women* by Laura I. Baldt. This study has not been published, but a summary of her findings is given by Charters.[6] In the entire text, there are 111 different mathematical concepts, 45 being geometrical, and the

TABLE I

ALGEBRAIC OPERATIONS FOUND IN FRESHMAN CHEMISTRY

First degree equation with one unknown	7
First degree equation with two unknowns	2
First degree equation with three unknowns	0
Second degree equation	0
Substitution in a formula	9
Algebraic addition	2
Algebraic subtraction	0
Algebraic multiplication	0
Algebraic division	0
Clearing of fractions	6
Simple ratio	18
Proportion	59

others arithmetical. The total number of appearances is 3,115. Of the concepts classified as geometrical, the seven occurring most frequently appear as follows:

TABLE II

GEOMETRICAL TERMS FOUND IN THE TEXT ENTITLED "CLOTHING FOR WOMEN"

Line	613
Equal	447
Center	311
Measure	285
Width	171
Length	132
Point	107

There are 430 "rhetorical equations" of the type: "*AE* equals one-sixth neck measure plus three-eighths inch." Besides requiring arithmetical computation, these equations involve:

1. Drawing a line parallel to a given line.
2. Drawing a line perpendicular to a given line.

[6] Charters, W. W., *Curriculum Construction*, pp. 241–43. The Macmillan Company, 1923.

3. Drawing a circle or an arc having given the center and the radius.
4. Reading geometrical figures by means of letters representing points.

Many studies have been made concerning the relation of high-school mathematics to high-school physics. Vira M. Crawford [7] made an analysis of four high-school texts. The text using the least mathematics mentioned per cents 63 times and proportions 165 times; it employed graphs 16 times. In the text having the fewest formulas, 28 algebraic formulas appeared, and in the one having the most 60 occurred. Geometrical terms appeared 1,731 times in the four texts. With the exception of the names of solids, these terms are from plane geometry. The sine is the only trigonometric function used.

The following tabulation shows the distribution of a few of the most frequently occurring mathematical expressions in the four textbooks examined by Miss Crawford, designated as A, B, C, D.

TABLE III

MATHEMATICAL EXPRESSIONS FOUND IN HIGH SCHOOL PHYSICS TEXTS

	A	B	C	D
Proportional	11	4	8	6
Inversely proportional	7	10	6	4
Directly proportional	8	11	11	4
Varies as	1	3	4	5
Varies inversely as	2	6	6	8
Formulas in the form of a proportion	8	8	13	11

Probably the most elaborate study of the mathematics needed in high school physics has been made by Kilzer [8] in connection with the making of an inventory test. After determining the five high-school texts most generally used in Iowa, Kilzer made a very careful analysis of these texts and their laboratory manuals, and then tabulated the frequency of occurrence of the various processes used in the expository material and the problems. He used three of the texts in making the count. The others he used merely to discover additional material. The eleven items which follow

[7] Crawford, Vira M., "The Mathematics Needed as a Prerequisite for the Study of High School Physics." Unpublished Master's Thesis, Colorado State Teachers College, June, 1927.

[8] Kilzer, Louis R., "The Mathematics Needed in High School Physics." Unpublished Doctor's thesis, State University of Iowa, 1928.

have been selected from the fourteen tables which appear in Kilzer's thesis.

1. The four fundamental operations occur 895 times in the three texts, but mostly with monomials. Division is with monomials only. The only multiplication with binomials is the square of the difference, which appears twice in each of two texts. A trinomial is multiplied by a monomial twice in each of the same two texts.

2. There are 95 cases in the three texts in which it is necessary for pupils to form equations.

3. Fractional equations occur 273 times in the three texts.

4. Quadratic equations occur 22 times in the three texts. There is no statement to indicate whether or not all of these are pure quadratics, but the resulting inventory test [9] has in it three pure quadratic equations to be solved and no affected quadratic equations.

5. Simultaneous equations are found three times.

6. There are 120 different formulas.

7. There are 278 cases where either transposition or clearing of fractions occurs in these formulas.

8. The three texts have a total of eight cases of factoring. Two are squares of the difference, and two are trinomials which are not perfect squares. The other four are more simple.

9. The process of completing the square occurs twice.

10. Under the heading "Axioms, Constructions, Measurements, Theorems, Corollaries Used in 5 Physics Textbooks," there is a total of 57 items. These are without exception the facts and processes which are emphasized in the first part of geometry. In addition to these items, the following formulas are listed: area of a circle; circumference of a circle; volume of a cube from lateral surface or from an edge; area of a cylinder; volume of a cylinder; area of a rectangle; volume of a rectangular solid; volume of a sphere.

11. The manuals require the following processes not found in the texts: compute value of pi; find reciprocal; find average; find "mean"; find per cent of error; construct graphs; explain use and give definition of *sine*.

[9] Kilzer, L. R. and Kirby, T. J., *Inventory Test for the Mathematics of High-School Physics.* Public School Publishing Company.

CHAPTER II

MATHEMATICAL VOCABULARY

For the compilation of the word list in this chapter the text written by Oscar M. Stewart [1] was used. Unlike the study by Williams which was reviewed in the first chapter, this list does not contain the mathematical terms which every child has repeatedly heard in the grades. Accordingly dates, denominate numbers, cardinal numbers, and ordinal numbers (included in the study made by Williams) have not been tabulated. Such well-known words as *add, product,* and *last* have not been included. The purpose is to present the mathematical terms which appear in physics, and which have been used so rarely prior to high-school work that their accurate meaning may not be known by the pupils. Words having a double meaning, such as *face* or *origin,* are counted only when they are used in a mathematical sense.

The words may be grouped into three general classes:

1. Words that are not legitimately a part of the vocabulary of high-school mathematics. Such words as *helix, ellipsoid, spiral, sinusoidal, angular deviation,* and *conjugate foci* belong in this group.

2. Words that are so commonly used in high-school mathematics that we may safely assume a satisfactory knowledge of their meanings. *Angle, point, center, surface,* and *sign* are illustrative of this group.

3. Words that appear frequently in this list, but are not sufficiently emphasized in all high-school classes and textbooks to give assurance that their meanings are thoroughly understood by the pupils.

The first two groups need not concern us further, but several words which evidently belong in the last group deserve special consideration.

The frequency of occurrence of such words as *axis, rotation, rotating, revolution, symmetrical, angular velocity,* and *angular*

[1] Stewart, Oscar M., *Physics, A Textbook for Colleges.* Ginn and Company, 1924.

TABLE IV

FREQUENCY OF MATHEMATICAL TERMS IN STEWART'S TEXTBOOK IN PHYSICS

WORD	FREQUENCY	WORD	FREQUENCY
accelerated	4	curvature	28
acceleration	70	curve	46
acute angle	1	curved	19
adjacent	1	cylinder	63
algebraic	9	cylindrical	4
algebraically	4	degree (s)	110
algebraic process	2	denominator	1
algebraic sum	2	depth	11
angle (s)	239	derivation	5
angular	48	derived	1
angular acceleration	2	deviation	17
angular deviation	1	diagonal	2
angular speed	14	diagram (s)	13
angular velocity	25	diagrammatically	9
arc (s)	16	diameter (s)	71
area (s)	51	directly proportional	7
average	60	diverge	11
axioms	2	divergence	8
axis	129	divergent	2
base	2	diverging	34
breadth	2	edge	10
broken line	8	eliminate	1
cancel	2	ellipsoid	3
cancelled	1	elongate	1
cancelling	1	elongation	8
center (s)	150	equal	457
central	2	equally	1
circle (s)	35	equate	1
circular	31	equating	2
circumference	4	equation	354
clockwise	10	equilateral	1
closed polygon	1	face (s)	15
coincide	2	factor (s)	20
concave	22	figure (s)	65
concentric	2	finite	4
conjugate	7	focal	65
conjugate foci	1	foci	12
constant	187	focus	85
converge	6	formula (s)	50
converged	5	geometrical	11
converging	36	geometrical sum	1
convex	25	graph	3
cosine	6	graphical	21
counter clockwise	11	graphically	8
cross section	26	height	40
cross sectional	3	helix	2
cube	3	horizontal	84
cubic	79	horizontally	4

TABLE IV (*Continued*)

Word	Frequency	Word	Frequency
hypotenuse	2	perpendicular	75
imaginary line	3	perpendicular line	2
incline	2	perpendicularly	9
inclined	5	plane (s)	118
increments	1	point (s)	315
indeterminate	1	polygon	3
infinite	5	positive	88
infinitely	1	positively	47
infinity	1	projected	3
integral	4	projection	4
intersect	3	proof	18
intersection	3	proportion	2
inverse	2	proportional	71
inversely	8	proportionally	2
inversely proportional	10	proposition	1
inverse proportion	1	prove	15
inverse ratio	1	proved	36
inverse square	1	prism	12
inversion	1	quadrant	2
invert	1	radian	15
inverted	8	radii	3
isosceles	1	radius	76
length	318	rate	145
line (s)	382	ratio	78
linear	21	rectangle	1
linear acceleration	1	rectangular	12
linear speed	3	rectilinear	3
linear velocity	5	right angle	45
longitudinal	15	right-angled triangle	2
mathematically	1	rotate	41
maximum	45	rotated	30
mean	3	rotating	45
minimum	16	rotation	88
minus	1	rotational	24
multiple	13	revolution	20
negative	125	revolve	2
negatively	42	revolving	3
normal	12	scalar	3
numerator	1	scale	56
numerical	24	sectional	1
numerically	13	semicircle	1
numerically equal	26	series	5
numerical value	26	side (s)	45
obliquely	7	sign	19
origin	2	similar	6
parabolic	4	similar triangles	3
parallel	164	sine	34
parallelogram	5	sinusoidal	2
per cent	58	solid	15
percentage	8	speed (s)	248

TABLE IV (*Continued*)

Word	Frequency	Word	Frequency
sphere (s)	20	transposing	1
spherical	29	transposition	1
spiral	9	transverse	19
square (s)	29	transversely	3
square (foot, etc.)	45	triangle (s)	29
square root	6	unification	1
straight	26	unequal	5
straight line (s)	32	unit (s)	424
subscripts	1	unity	24
subtends	1	variables	1
surface (s)	234	varies	2
symbol	4	varies inversely	19
symbolically	1	vector (s)	79
symmetrical	10	velocity	96
symmetry	8	vertical	67
tangent	19	vertically	6
tangential	5	volume (s)	156
tangentially	5	width	3
theorem	12		

speed suggests a group of related concepts in physics which might receive much more emphasis in high-school mathematics. It is true that the terms *axis* and *revolution* appear in the study of the cone of revolution, and the cylinder of revolution, and that the idea of rotation may be used incidentally in other connections, but if teachers realized the value of the ready use of these terms, they might easily find other opportunities to introduce them even in the junior high school. Many of the junior high-school pupils of course will not attend college, but this sort of training is of sufficient value in itself to justify its introduction in the junior high school. The rotation of planes about axes might be introduced in informal discussions. The study of crystallography in chemistry makes a definite demand for these concepts. The study of symmetry has not been given a prominent place in geometry in the past, probably because Euclid did not refer to it. Its possibilities in the junior high school are discussed by Smith and Reeve.[2] Even so elementary a discussion as the one recommended here for the junior high school would be superior to that formerly given in some senior high schools. A quotation from *The Teach-*

[2] Smith, David Eugene and Reeve, William David, *The Teaching of Junior High School Mathematics*, pp. 149–51. Ginn and Company, 1927.

ing of Geometry in Schools[3] indicates the attitude of the British Mathematical Association:

> Symmetry should have a larger place in teaching than has been usual. It is a conspicuous quality both of the things around us and of many geometrical figures with which we deal. It is not less worthy of notice and study than other properties such as congruence and similarity. . . . It would be easy to draft a rigorous set of definitions and theorems on symmetry, but this is not the best way of presenting the subject in elementary teaching. What should rather be cultivated is a *sense* for symmetry. . . . The main point is that the idea itself becomes familiar and useful. . . . Line symmetry may be used as a means of approach to certain facts: for example, the properties of an isosceles triangle, the two standard locus theorems, the relation of the diameter to a chord of a circle, as well as other properties of the circle and those of tangents. Similarly the corresponding facts in solid geometry may be approached by plane symmetry. Point symmetry may be introduced as the property of a figure such that the line joining any pair of corresponding points is bisected by the center. . . . Point symmetry affords a convenient summary of the properties of the parallelogram.[4]

Ample opportunity for the presentation of *angular velocity, rotation about a point, radian measure, clockwise* and *counter-clockwise motion* is offered in the study of trigonometry. The three trigonometric functions, *sine, cosine,* and *tangent* are indispensable in the study of physics. An elementary knowledge of trigonometry is assumed in college freshman physics. Of the terms here listed, the concept of *radian* is probably the most troublesome. The double use of π as the ratio of the circumference to the diameter of a circle, and as the unit in circular measure causes confusion. One contributing factor to this confusion is doubtless the custom of neglecting to state the unit of measure when angles are expressed in radians. Angular velocity is not given so much consideration in trigonometry as its extensive use in the study of rotary motion would seem to justify. *Clockwise* and *counter-clockwise motion* can and should be introduced at an early stage in mathematics in order that pupils may become thoroughly familiar with the concept.

A formal presentation of *maxima* and *minima* is not necessary in order to prepare pupils for the use of these concepts as found

[3] *The Teaching of Geometry in Schools, Mathematical Association,* pp. 48–50. G. Bell and Sons, London, 1923.

[4] Teachers interested in the subject of symmetry should consult the following reference: Gugle, Marie, "Dynamic Symmetry," *Third Yearbook of the National Council of Teachers of Mathematics,* pp. 57–64. Bureau of Publications, Teachers College, Columbia University, 1928.

in the study of college subjects. There are many opportunities to make use of the words *maximum* and *minimum* in informal discussions of the relationship of the sides of a *hinged parallelogram* in order that it may have the *maximum area*; the form of a *rectangular quadrilateral* in order that it may have the *minimum perimeter* for a stated area; the *maximum* and *minimum* value of the *sine* and of the *cosine* as shown by their curves. The last may be so taught that it will provide an excellent background for simple harmonic motion, or it may be so neglected as to handicap the pupil when he later meets that subject.

The words *positive* and *negative* appear more than two hundred times. There is no need of further emphasis of the symbols as they appear in algebra, but the oppositeness of positive and negative as qualities or characteristics is first met in mathematics. It is this interpretation that must be emphasized until it carries over readily to positive and negative poles, positive and negative charge, positive and negative ionization, positive and negative correlation, and the various other applications of the terms.

Such expressions as *constant speed, constant volume, constant pressure,* and *constant temperature* usually involve the consideration of something else that is not constant. The opportunities of the teacher of mathematics to prepare pupils for this relationship are numerous: an angle inscribed in a given segment of a circle is constant; the sum of the squares of the segments of two chords intersecting at right angles within a circle is constant, even though their lengths vary; the equation of a circle is $x^2 + y^2 = r^2$ (a constant); the constant in the general quadratic equation is essential in determining the nature of the roots. The manner in which variables and constants are interrelated in physics and chemistry would be better understood if they were discussed more in mathematics.

The idea of a *normal* to a curve or to a surface is first met by many students when they study analytic geometry, unless they have previously used the term in physics. There can be no objection to the earlier introduction of such a term when it is definitely known that the term will be used later in the same sense in both mathematics and physics.

Such words as *converge, divergent,* and *converging* occur approximately one hundred times. Some geometry texts discuss pencils of lines, converging and diverging. Many texts do not.

Such discussions if not too formal prepare the pupil for future work in both mathematics and physics.

The ideas suggested by the words *cross section, longitudinal,* and *transverse* occur frequently in physics. If these expressions are used freely in the work on solids, students from such classes will have an advantage over their companions in their college work.

Many laws and formulas in physics concern quantities which are *numerically equal.* In mathematics, *numerically equal* refers to quantities which may be either alike or unlike in algebraic signs and have the same *absolute* value. By the use of the C. G. S. (centimeter gram second) system, it is possible to combine various numerical values expressed in different units of measure in one formula and to place the result *numerically equal* to some other quantity. This double use of *numerically equal* can be illustrated and made clear to high-school pupils.

The frequent use of the expressions *inverse, inversely proportional, varies inversely, directly proportional,* and *ratio* suggests the one series of mathematical concepts in physics for which students are probably the most poorly prepared. In referring to students in physics, Young [5] makes the following statement:

> Of the topics of algebra, the pupils are found to be especially weak in interpreting formulas, in passing from literal to numerical expressions, in interpreting equations, in applying variation (direct and inverse), in ratio and proportion. This gives the teachers of mathematics the cue to pay special attention to these topics. The last three of these topics really fall under fractions and should be treated in a simple straightforward manner, without a mass of terms like extremes and means, antecedents and consequents, and rules involving these terms. No theory whatever is needed except the plain theory of fractions. These topics all lend themselves especially well to concrete treatment, and may be made among the most interesting of the whole subject, while at the same time preparing pupils to meet the justifiable expectation of teachers of physics, that pupils coming to them be readily able to answer such questions as:
>
> $$\text{If } t = 2\pi\sqrt{\frac{l}{g}} \text{ how does } t \text{ vary with } l? \text{ with } g? \text{ etc.}$$

It is interesting to note that this identical statement appears on the same page of Young's book in the first edition published in 1907. The truth of the statement and the need of the admonition to abolish such terms as *extremes, means, antecedents,* and *consequents* are still evident.

[5] Young, J. W. A., *The Teaching of Mathematics,* p. 323. Longmans, Green & Company, 1927.

The following are among other expressions found in physics that suggest concepts that might be strengthened by the proper presentation in high-school mathematics: *center of curvature, curved line tangent to a circle, parallels of latitude* (small circles parallel to great circles), *a plane determined by two lines,* either parallel or intersecting, *directed lines,* and a point on each of two lines being at their point of intersection.

If teachers of high-school mathematics will carefully check the words and phrases here listed, each one will probably find fifteen or twenty items (not necessarily the same ones) which he can teach more effectively if in presenting them he is mindful of the fact that in the future they are to function in the study of some college subject.

CHAPTER III

SYMBOLISM

Mathematics is a language. Its power to express facts succinctly and without ambiguity is due to its symbolism. Just to the extent that this symbolism is universally accepted, are we able by means of mathematics to have a universal language. The purpose of this chapter is to show how other subjects make use of symbolic language. Table V shows how certain texts in college physics have extended the standard symbolism of mathematics. The standard symbols of elementary mathematics for *plus, minus, times, divided by, equals, integral roots, integral powers,* and *parentheses* are not included in the tabulation. These are used uniformly in other subjects as occasion demands. The following texts were examined:

A. Stewart, Oscar M., *Physics, A Textbook for Colleges.* Ginn and Company, 1924.

B. Kimball, Arthur L., *A College Textbook of Physics.* Henry Holt and Company, 1917.

C. Webster, David Locke, Farwell, Hermon W., and Drew, Elmer Reginald, *General Physics for Colleges.* The Century Company, 1926.

D. Smith, Alpheus W., *The Elements of Physics.* McGraw-Hill Book Company, 1927.

E. Spinney, Louis Bevier, *A Textbook of Physics.* The Macmillan Company, 1925.

In addition to the symbols listed in Table V, various letters are used in one or more textbooks without any apparent uniformity, such as the following: α for a specific amount of rotation, α for angular acceleration, ρ for the density of a medium, ρ for resistivity, i for current, i for angle of incidence, τ for interval of time, δ for density of air, ϕ for flux, β for coefficient of cubical expansion, μ for index of refraction, e for fundamental unit of electricity. One textbook uses e in three different senses as follows: fundamental unit of electricity, modulus of elasticity of a medium, and base in Napierian logarithms. Another text uses μ for co-

TABLE V

Symbols Used in College Physics Textbooks

Description of Symbol	Symbol	Textbook				
		A	B	C	D	E
Capital letter for a point	A	√	√	√	√	√
Small letter for a point	a		√			
Two capital letters for a line or quantity	AB	√	√	√		√
Two small letters for a line or quantity	ab		√		√	
One capital letter for a line or quantity	A	√	√	√	√	√
One small letter for a line or quantity	a	√	√	√	√	√
One capital and one small letter for a line	Ab		√			
One capital letter in the double sense of a point and a quantity	A		√			
One small letter in the double sense of a point and a quantity	a		√			
One small letter for an angle	a	√	√	√	√	√
One capital letter for an angle	A				√	
Three capital letters for an angle	ABC	√	√			√
Three capital letters for the angle formed by the line A meeting the line B at O	AOB					√
One Greek letter for an angle	ϕ	√		√	√	√
Three small letters for a triangle whose sides are p, w, and r	pwr					√
The ordinary angle sign	$\angle$	√				
The ordinary angle sign with an arc	$\measuredangle$					√
Numerical subscript	V_3	√	√	√	√	√
Literal subscript	R_p	√		√	√	√
Prime	v'	√	√	√	√	√
Double prime	v''	√	√	√	√	√
Small Roman numeral for fourth prime	a^{IV}					√
A subscript and prime with one letter	V_3'			√	√	
A subscript and exponent with one letter	V_3^2	√	√	√	√	√
Fractional exponent	$a^{\frac{1}{2}}$		√	√	√	

Note: A check (√) under the letter A, B, C, D, or E means that the symbol described at the left appears in the book indicated. The list of books is given on page 15.

TABLE V (*Continued*)

Description of Symbol	Symbol	Textbook				
		A	B	C	D	E
Literal exponent	k^p			√		
Double colon in a proportion	::		√		√	
Colon between the terms of a ratio	:	√	√	√	√	√
Pi for ratio of circumference to diameter	π	√	√	√	√	√
Infinity sign	∞		√		√	√
Standard sign for *varies as*	$\propto$	√	√	√		√
Standard sign for *greater than*	$>$		√	√		√
Standard sign for *less than*	$<$		√	√		√
Standard form number*	2.7×10^8	√	√	√	√	√
Plus and minus for poles of battery or magnet	$+-$	√	√	√	√	√
The equality sign in chemical equations	$=$		√		√	√
Sigma for summation	Σ					√
The dot between quantities for *times*.	.					√
Slanting line to divide the terms of a fraction	P'/P	√		√	√	√
Usual abbreviation for sine	sin	√	√	√	√	√
Usual abbreviation for cosine	cos	√	√			√
Usual abbreviation for tangent	tan	√	√			√
Two small letters within parentheses with *sin* to indicate sine of the angle between them	sin (l,r)			√		
A capital and a small letter to indicate two values of the same kind of thing	M,m	√	√	√	√	√
Greek letter omega for angular velocity	ω	√	√	√	√	√
Greek letter lambda for wave length.	λ	√	√	√	√	√
Greek letter mu for permeability	μ	√	√	√	√	√
Subscripts in chemical formulas	H_2SO_4		√	√	√	√
A bar over a letter to indicate *average*	$\overline{v}$	√				√
A slanting line separating ft. from sec. with an exponent 2 to indicate "feet per second per second"	ft./sec.^{2}	√	√	√	√	

* "A number is expressed in the standard form when it is written as a decimal with a single digit before the decimal point multiplied by some positive or negative power of ten." Nunn, T. Percy, *The Teaching of Algebra (Including Trigonometry)*, p. 175. Longmans, Green & Company, 1927.

TABLE V (*Continued*)

Description of Symbol	Symbol	Textbook				
		A	B	C	D	E
A slanting line separating gm. from cm. with an exponent 3 to indicate *grams per cubic centimeter*.....	gm./cm.^{3}	√	√			
A slanting line separating lb. from in. with an exponent 2 to indicate *pounds of pressure per square inch*.	lb./in.^{2}	√	√	√		
The same abbreviations as above with a horizontal line instead of a slanting line	$\frac{\text{ft.}}{\text{sec.}^2}$			√		√
The horizontal lines separating miles, hour, minute, to indicate miles per hour per minute.....	2 $\frac{\text{miles}}{\frac{\text{hour}}{\text{minute}}}$					√
A negative one above and to the right of sec. to indicate *vibrations per second*	2,906 sec.$^{-1}$	√				

efficient of friction, for index of refraction, and for micron. The same text uses the regular infinity sign seven times to mean *varies as,* and the standard sign twelve times for *varies as*. The initial letter with a subscript zero is used rather indiscriminately in the various texts. The symbol v_0 is used for velocity of light in a vacuum, initial velocity, velocity of sound waves at 0° centigrade, volume at 0° centigrade, and initial volume; and d_0 is used for initial density and for density of air at standard atmospheric pressure. Other letters used thus are T_0 for lowest temperature of a heat engine, I_0 for intensity of illumination at unit distance, R_0 for resistance of a conductor at 0° centigrade, p_0 for pressure at 0° centigrade, l_0 for initial length, and h_0 for initial height.

The tabulation here given does not exhaust the list of symbols used in these five texts, but it is sufficient to show the great variety and the lack of uniformity and agreement among the various textbooks on the subject.

It is evident that the customary method of lettering figures in geometry is not followed elsewhere. Every possible combination of

capital and small letters for indicating straight lines has been noted. It seems rather unusual to a teacher of high-school mathematics to speak of a point b, a line bc, a line Ba, a triangle pwr, sin (l,r), or an angle AOB formed by a line A meeting a line B at the point O. There is something that seems inconsistent in interpreting ft./sec.^{2} as "feet per second per second," and then interpreting lb./in.^{2} as "pounds per square inch." One text [1] explains that "The term *second*^{2} is an abbreviation for *second* $\times$ *second* just as *ft.*^{2} is often used for a unit of area as an abbreviation for *ft.* $\times$ *ft.* or *sq. ft.*" To interpret 2906 sec.$^{-1}$ as *2906 vibrations per second* is a purely arbitrary convention which is unfortunate since a negative 1 written above and to the right has two definite and well-established meanings in mathematics.

The promiscuous use of capital and small letters and Greek letters with subscripts and primes, the same letter frequently having both a subscript and a prime, or the same letter having a subscript and an exponent, presents a situation which many students have not met in their high-school courses in mathematics.

This discussion of the symbolism in physics is not presented for the purpose of persuading the writers of college texts in physics to adopt a uniform and consistent set of symbols, much as that may be needed. Nor is it intended to convince teachers of high-school mathematics that they should attempt to prepare their pupils for college physics by teaching these symbols. It is merely to call attention to a real difficulty which exists and which has been given very little consideration, if any, by either the college teacher of physics or the high-school teacher of mathematics.

Chemistry probably furnishes the best illustration of the possibilities of symbolic language. In Table VI are shown the symbols which are used in the following five texts:

A. Holmes, Harry N., *General Chemistry.* The Macmillan Company, 1925.
B. McPherson, William, and Henderson, William Edwards, *A Course in General Chemistry,* Third Edition. Ginn and Company, 1927.
C. Smith, Alexander, *College Chemistry,* Revised and Rewritten by James Kendall. The Century Co., 1923.
D. Foster, William, *The Elements of Chemistry.* D. Van Nostrand Company, Inc., 1925.
E. Brinkley, Stuart R., *Principles of General Chemistry.* The Macmillan Company, 1926.

[1] Webster, Farwell, Drew, *op. cit.*, p. 89.

TABLE VI

SYMBOLS USED IN COLLEGE CHEMISTRY TEXTBOOKS

DESCRIPTION OF SYMBOL	SYMBOL	TEXTBOOK				
		A	B	C	D	E
Arrow used in place of the equality sign in chemical "equations"	$\rightarrow$	√	√	√	√	√
Double arrows to indicate that the chemical reaction is reversible	$\rightleftarrows$	√	√	√	√	√
Arrow pointing upward to indicate that a gas results from the reaction	$\uparrow$	√		√	√	√
Arrow pointing downward to indicate that a precipitate results from the reaction	$\downarrow$	√		√	√	√
Subscripts in the usual sense to indicate two values of the same quantity	v_1, v_2	√	√	√	√	√
Subscripts to indicate the number of atoms	H_2SO_4	√	√	√	√	√
Parentheses with subscript to indicate the number of radicals	$Ca(OH)_2$	√	√	√	√	√
Literal subscript in usual sense	V_s		√	√		
Literal subscripts in general chemical formula	C_nH_{2n+2}	√	√	√	√	√
Prime	T'	√	√			
Double prime	T''		√			
Small Roman numerals in position of primes after symbols of elements, for valence	C^{IV}			√	√	
Small x for multiplication	x	√	√			√
Dot above the base line for multiplication	$\cdot$			√		
Small Arabic numeral immediately over the symbol of an element to indicate valence	$\overset{4}{C}$		√			
One or more small plus signs in the position of an exponent after a symbol for an element to indicate one or more positive charges to an ion	C^{++}		√	√	√	√
One or more small minus signs in the position of an exponent after a symbol for an element to indicate one or more negative charges to an ion	Cl^-		√	√	√	√

NOTE: A check (√) under the letter A, B, C, D, or E means that the symbol described at the left appears in the book indicated.

TABLE VI (*Continued*)

Description of Symbol	Symbol	Textbook				
		A	B	C	D	E
One or more small plus signs immediately above the chemical symbol to indicate one or more positive charges to an ion	Ca^{++}	√				
One or more small minus signs immediately above the chemical symbol to indicate one or more negative charges to an ion	Cl^{-}	√				
A small zero in the position of an exponent after the chemical symbol to indicate that it has neither a positive nor a negative charge	Zn°			√		√
Brackets inclosing a chemical symbol to indicate molecular concentration as used in the mass action formula	[H]	√	√	√		√
The mathematics sign for infinity used for "varies as"	∞	√				
The Greek letter theta to represent the quantity of electricity carried by one equivalent of a negative ion	θ			√	√	
Double colon between the ratios forming a proportion	::	√		√		
Standard form number to express very small or very large quantity	1.7×10^{-8}	√	√	√	√	√
A small digit following the first zero after a decimal point to indicate the number of zeros omitted (used in tables)	$.0_3356$	√		√		√

From a study of this tabulation, it is evident that the student of chemistry must appreciate the power of a symbolic language in expressing the various chemical reactions and relationships. When several of these symbols are combined into one formula, the amount of information imparted by a relatively simple statement is illustrated by the following:

$$Zn + 2HCL \rightarrow ZnCl_2 + H_2\uparrow$$

This one formula expresses in a condensed form the following facts which have been ascertained by laboratory investigation:

1. One molecule of zinc reacts in solution with two molecules of hydrochloric acid (which consists of one atom of hydrogen and one atom of chlorine) to yield one molecule of zinc chloride (consisting of one atom of zinc and two atoms of chlorine) and one molecule of hydrogen (consisting of two atoms) which is freed as a gas.
2. The reaction is not reversible.
3. Zinc is more active than hydrogen, since it displaces the hydrogen.
4. The valence of zinc is 2, since it combines with 2 atoms of chlorine.
5. From the atomic weights, 65.4 grams of zinc react with a compound containing 2 grams of hydrogen (approximately) and 71 grams of chlorine to form 136.4 grams of soluble zinc chloride, and to liberate 2 grams of hydrogen (approximately) which will make 22.4 liters of hydrogen gas.

This study concerns itself primarily with college freshman subjects, but there is one subject not offered in the freshman year which should be considered. The study of the theory of statistics has probably increased more rapidly in the past ten years than the study of any other college subject. It is being studied by thousands of students in junior, senior, and graduate courses, who, unfortunately, have had but little, if any, college mathematics. This may not be as it should be, but conditions must be met as they are. A thorough understanding of the underlying principles of statistics involves much more mathematics than can be taught in the high school, but the symbolic language employed in the study of statistics is easily within the grasp of the high-school graduate.

Because of the recent rapid development of statistics as now taught, it is inevitable that there should be a diversity of symbolism. The study of statistics is preëminently a study of mathematical relations, and as such involves the standard mathematical symbols. In addition to these standard symbols, many new ones have been devised. A complete tabulation of the symbols used in representative texts would merely give further emphasis to the importance of a symbolic language in any science dealing with quantitative data. The ability of the student to adapt himself to the specific symbolic language of the author being studied is

probably more essential in statistics than in any other subject. Holzinger,[2] for example, gives a list of 133 symbols used in his text. Yet a careful examination of fifteen texts—five in educational statistics, five in economic statistics, and five in mathematical statistics—reveals no inconsistencies in the use of established mathematical terms and symbols. The differences in symbols occur in connection with the newer statistical terms such as partial correlation, multiple correlation, and regression (or predicted) y.

The repeated and varied uses of subscripts, both literal and numerical, with capital letters, small letters, and Greek letters, suggest that the student who has had experience in using subscripts will have a distinct advantage. Primes are also used by many of the texts to have a specific meaning. Thus x' is used by at least three of these texts to signify deviation from an assumed mean to distinguish from the deviation from the true mean which is indicated by the x without the prime.

There is one standard mathematical symbol which appears frequently in every one of the fifteen statistics texts examined, and in one of the five physics textbooks discussed in the first part of this chapter, but which is not ordinarily taught in high-school mathematics. Capital sigma is used uniformly as the standard sign for summation. Thus ΣX means *the sum of the scores;* Σx means *the sum of the deviations from the mean;* and $\Sigma x'$ means *the sum of the deviations from the assumed mean.* The letter for scores or for deviations may vary in the different texts, but the mathematical symbol for summation is used without variation.

Teachers of high-school mathematics and authors of high-school texts in mathematics are fully aware of the fact that the symbols used in other subjects differ materially from those ordinarily appearing in mathematics texts. It is evident, however, that the assumption has been made that if the pupil has been taught the operation with the ordinary letters, a, b, c, x, y, and z he will be able to transfer to the same operation with subscripts, capitals, and Greek letters. Thorndike[3] reports an interesting

[2] Holzinger, Karl J., *Statistical Methods for Students in Education,* pp. 360–65. Ginn and Company, 1928.

[3] Thorndike, E. L., "The Effect of Changed Data Upon Reasoning," *Journal of Experimental Psychology,* Vol. V, No. 1, pp. 33–38 (February, 1922).

experiment that bears directly upon this question. The following nine problems were arranged in the customary or "habitual" form, and then again in the changed form:

Habitual Form	Changed Form
1. What is the square of $x + y$?	What is the square of $b_1 + b_2$?
2. What is the square of a^2x^3 ?	What is the square of $r_1^3\, r_{11}^2$?
3. Simplify $4ac + \left(\frac{b^3}{c^3} \times \frac{c}{d^3} \times \frac{d^3}{b}\right)$	Simplify $5a^2b + (b^3 \div c^3)(c \div d^3)(d^3 \div b)$
4. What are the factors of $x^2 - y^2$?	What are the factors of $\frac{1}{x^2} - \frac{1}{y^2}$?
5. Multiply x^a by x^b	Multiply 4^a by 4^b
6. Simplify $ac - [a(b + c)]$	Simplify $p_1p_2 - [p_1(p_2 - p_3)]$
7. Solve $x + y + z = 15$, $2x + y + 3z = 22$, $x + 2y + z = 25$	Solve $c_1 + c_2 + c_3 = 15$, $2c_1 + c_2 - 3c_3 = 22$, $c_1 + 2c_2 + c_3 = 25$
8. $e^2 + ef = \frac{g}{x}$ What does x equal?	$e^2 + ef = \frac{g}{p}$ What does p equal?
9. There are two numbers. The first number plus 3 times the second number equals 7. The first number plus 5 times the second number equals 11. What are the numbers?	$y = ax + b$ When $x = 3$, $y = 7$ When $x = 5$, $y = 11$ What does a equal? What does b equal?

Thorndike rearranged these eighteen problems with various other problems in order to camouflage the purpose of the test, and then gave certain ones to each of three groups of students in such a manner that he could compare the results on the problems in the habitual form with those in the changed form although no student realized that he was working two forms of the same problem. There was a total of 97 graduate students in the experiment. The per cents wrong or incomplete were shown by Thorndike in Table VII.

TABLE VII

PER CENT OF PROBLEMS WRONG OR INCOMPLETE *

PROBLEM	HABITUAL FORM	CHANGED FORM
1	6%	28%
2	34	47
3	37½	64½
4	22	41
5	55	70½
6	25	53
7	62	61
8	52	53
9	16	70

* See Thorndike, *op. cit.*

The report is concluded with the following statement:

On the whole, it seems certain that even such slight changes as from customary a, b, x, and y to k and p or to p_1, p_2 and p_3 or p_1, p_{11}, and p_{111} impede thought, and that the general theorem does hold that *any disturbance whatsoever in the concrete particulars reasoned about will interfere with reasoning.*

A recent experience of approximately one hundred graduate students in statistics furnishes an interesting illustration in the changing of symbolism. These students had been working for two weeks on the subject of partial correlation. Numerical subscripts had been used for the partial sigmas, and the partial and multiple correlation coefficients, with occasional reference to literal subscripts. At the end of the two weeks, a test was given in which the initial letters of the four traits considered were used as subscripts instead of the customary numerical subscripts. R was used for reading score, A for arithmetic score, M for mental age, and C for chronological age. The confusion of these adult students in changing from numerical to literal subscripts was quite evident. They were later asked to report on the following questions:

1. "How many in writing out the required formulas with literal subscripts found it necessary to think of these formulas in terms of numerical subscripts as a preliminary step or as a means of checking results?"

2. "How many were able to write out the formulas with lit-

eral subscripts immediately without any thought of the numerical subscripts?"

The replies showed that 59 thought they were dependent upon the patterns with numerical subscripts, and that 39 thought they had transferred directly to the literal subscripts.

The formulas referred to in the preceding paragraph have in them a new symbolism—the use of the dot to separate the subscripts representing traits which are held constant from the subscripts for the remaining traits or trait. The expansion for the formulas in four variables for the regression equations, partial sigmas, partial correlation coefficients, and multiple correlation coefficients had previously been made and practiced upon with numerical subscripts. These formulas were new and complicated. They were learned mechanically, without a clear concept of their significance or origin. When these students encountered the new situation calling for literal subscripts, the transfer was not readily made. This case is comparable to that of the high-school graduate who, after he has learned by a more or less mechanical process that if $a = 3\frac{x}{7}$ then $x = \frac{7a}{3}$ meets the problem: "Find ϕ if $Q = N_2 \frac{\phi}{R_2}$."

There are perhaps between three hundred and four hundred different symbols used in physics, chemistry, and statistics. Nobody would advocate the teaching of these symbols in high-school mathematics. Out of the entire number, there is one symbol not usually seen in high-school textbooks, the teaching of which may be justified. The summation sign, Σ, has been mentioned as occurring in one physics text and in all of the statistics texts. Students in college mathematics use it constantly. There is nothing so mysterious about it that it cannot be understood by high-school pupils, and yet students in statistics who meet it for the first time are confused by its use. The average class grade can be computed from the formula $A = \frac{\Sigma g}{N}$ where A stands for average, g stands for grade, and N stands for the number of pupils in the class. If a_1, a_2, and a_3 are the angles of a triangle, then $\Sigma a = 180°$. Some teachers use the summation sign in connection with arithmetic and geometric series. Students gladly accept any system of shorthand that saves work. The summation sign can thus be introduced in the high school with no difficulty whatever, and with a very definite advantage to those who will use it later.

In the preceding illustration, subscripts, too, are used as they will be used in later courses in college.

Many high-school pupils have been trained to number the angles of their geometric figures, and to refer to them as $\angle 1$, $\angle 2$, and so on. This is a quick, convenient means of referring to angles and it is in general use. There are excellent authorities, however, who oppose this method of referring to angles. The following statement in the National Committee Report [4] doubtless refers to angles as well as other magnitudes:

> In general, it is recommended that a single letter be used to designate any geometric magnitude whenever there is no danger of ambiguity. The use of numbers alone to designate magnitudes should be avoided by the use of such forms as A_1, A_2.

In the texts examined, angles are indicated by capital letters, small letters, Greek letters, in fact in practically every imaginable way except by number. Not a single case is found where an angle is numbered. This is another argument against the numbering of angles. The Greek letter or single small letter predominates in the texts studied.

As a general principle, it seems safe to state that when two symbols for the same thing are used in mathematics, the one which is found more frequently in other subjects should have the preference unless there is some well-founded objection to it. The use of small letters or Greek letters rather than numbers to designate angles illustrates this principle. The application of this principle to the use of the double colon in a proportion a few years ago might have left us in a quandary. At the present time, however, there is no doubt. The majority of the college texts are in agreement with the recommendation of the National Committee [5] that the double colon be abandoned. This does not necessarily mean that the high-school teacher should ignore the double colon entirely. He may be teaching from a textbook that uses it. In this case, the equality sign should be introduced with the explanation that it is preferable although the double colon is not absolutely wrong. If the text makes use of the equality sign, it might even then be well to mention the double colon as a sign that frequently appears in older texts. Two of the five physics

[4] National Committee Report, p. 79.
[5] *Ibid.*, p. 82.

texts and two of the chemistry texts which were examined still use the double colon even though they are quite recent editions.

The same paragraph in the National Committee Report that recommends the abandoning of the double colon mentions the symbol for *varies as* as "practically obsolete." Four of the five physics texts use it, and one of the chemistry texts uses the infinity sign in place of the standard sign for *varies as*. It may be obsolete in mathematics, but it cannot be so classified in science. It must be admitted that most of the writers of the present college texts studied their high-school mathematics in a previous generation, and yet the fact remains that these are the texts that our high-school graduates of to-day must study. Regardless of the form in the high-school text being studied, it will do no harm for the teachers to point out the three ways of stating the fact that the area of a circle varies as or is proportional to the square of the radius—$A \propto r^2$, $A = \pi r^2$, or $\frac{A_1}{A} = \frac{R^2}{R^2_1}$.

A later report[6] than that of the National Committee was recently prepared by a Sectional Committee on Scientific and Engineering Symbols and Abbreviations. This report has been generally approved by the scientific and mathematical organizations. The use of the symbol for *varies as* is approved as a standard symbol by this committee. This report should doubtless be accepted as superseding the recommendation of the National Committee which appeared five years earlier. In reference to proportion, this committee goes further than the National Committee by saying, "Use $a/b = c/d$, and discourage the use of $a : b :: c : d$." It is thus seen that this committee objects also to the use of the colon. It accepts the three forms a/b, $a \div b$, and $\frac{a}{b}$ for division. In connection with the discussion of division symbols, it points out that $a - b/c - d$ should not be used for $(a - b)/(c - d)$. The device used in the tables of several of the chemistry texts to indicate the omission of zeros at the right of the decimal point is approved as a standard method. Thus 0.0^3314[7] means 0.000314,

[6] "American Standard Mathematical Symbols," *The American Mathematical Monthly*, Vol. XXXV, No. 6, pp. 300–4 (June–July, 1928).

[7] According to the report in *The American Mathematical Monthly*, the small digit is in the position of an exponent as shown in the illustration, but in the three texts in chemistry in which this device is used, it is in the position of a subscript.

$0.0^{4}76$ means 0.000076. The following distinction is made between symbols and abbreviations:

> It is desirable to distinguish between (1) a *symbol* that is a single letter or single letter affected with subscripts, etc., which is to be used to represent a numerical value in a formula; and (2) an *abbreviation* which may consist of several letters but is not to be substituted for a numerical quantity in a formula.

The chapter on "Terms and Symbols in Elementary Mathematics" in the National Committee Report [8] gives an excellent summary of the symbols that have become obsolete or that are local in their use. It is recommended that the use of such symbols be discouraged. So far as the use of these symbols in publications is concerned, this point of view is absolutely justifiable. On the other hand, there is a necessity of pointing out to pupils that symbols are merely conventions, and not universal truths. When there is a question concerning a choice between two symbols for the same thing, the one most generally used should be adopted, but this does not prevent a local group from agreeing upon significant symbols to illustrate the possibilities of symbolic language. In a geometry class, for example, the pupils will readily accept the idea of placing equality signs on the given equal lines, and question marks on the parts that are to be proved equal. A question mark placed over the equality sign in a trigonometric identity to be proved is significant, but not universally used. There is no limit to the possibilities of local devices for illustrating the power of symbolic language, and for interesting pupils in this shorthand method of expressing themselves. The student beginning chemistry, physics, statistics, or any other subject dealing with quantitative data has a distinct advantage if he recognizes the possibilities of symbols, and has had some experience with them.

The use of a capital letter and a small letter to represent two values, such as M and m for two masses, R and r for two resistances, or I and i for two measures of currents, is very common in many of the texts. This is contrary to the general convention in high-school mathematics that a small letter represents a quantity, and a capital letter represents a location or point. The pupils should be made to understand that this is merely a convention. There are many opportunities to use initial letters, large and small, in the same formula in geometry. The area of the annulus

[8] National Committee Report, pp. 74–85.

is the area of the larger of two concentric circles minus the area of the smaller circle. The formula can be written

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

The fact that the circumferences of two circles have the same ratio as their radii can be stated $C/c = R/r$. Much can be said in favor of using primes and subscripts rather than capital and small letters, but there are cases, such as the above, where the capital and small letters are significant, and should therefore be used in order to familiarize the pupils with a symbolism that is frequently employed in other subjects. In cases where the letters do not definitely represent larger and smaller values it is possible that the primes and subscripts are preferable.

Thorndike[9] makes the following very significant statement concerning the use of symbols:

> Bonds should be formed with tasks extending algebraic symbolism to include Roman letters, capital letters, primes, and subscripts, and using other letters than *a, b, c, x, y,* and *z*. This is of real though not of great importance. There are three reasons for it. First, the generality of algebraic symbolism is more surely taught. Second, the fact that any literal number does in actual use stand for some number of something is more surely kept in mind if we let n_1 and n_2, rather than x and y, equal the smaller number and the larger number, if we let l', l'', and l''', or p_1, p_2, and p_3 equal the lengths of the three pendulums in inches, etc. Third, the pupil is better fitted for reading algebraic discussions. Customary usages should also be followed in the symbols for angles, masses, distances, times, etc., other things being equal.

[9] Thorndike, E. L. and others, *The Psychology of Algebra,* p. 252. The Macmillan Company, 1923.

CHAPTER IV

FACTS, CONCEPTS, AND SKILLS

The greater part of this chapter is devoted to the tabulation and discussion of the facts, concepts, and skills involved in reading the 707 pages and in solving the 572 problems in the text by Oscar M. Stewart.[1]

In the tabulation of the problems, if there were two distinct subheads requiring separate solutions, they were counted as two problems even though they were under one number. Whenever there were two or more apparent methods of solution, if one of these methods had been previously used and tabulated, it was repeated. If the possible methods were new, then the one which involved the simpler mathematics was used. The following problem (Number 17, p. 462) illustrates this method of choice:

PROBLEM

A certain ammeter has a resistance of 0.1 ohm. It is desired to add a shunt to it so that the ammeter proper and the shunt will together carry a current a hundred times greater than that indicated on the ammeter scale. What must be the resistance of the shunt?

SOLUTION

Let R and I represent the resistance and current respectively of the ammeter, and r and i represent the resistance and current of the shunt.

Formula: $\frac{I}{i} = \frac{R}{r}$. Given: $R = 0.1$ and $\frac{I+i}{i} = 100$.

According to the *model solution* on p. 451, the theorem "If four quantities are in proportion, they are in proportion by composition" is used. The solution is therefore as follows:

$$\frac{I+i}{i} = \frac{R+r}{r}.$$

Substituting:

$$100 = \frac{0.1+r}{r}$$

$$99r = 0.1$$

$$r = 0.00101.$$

[1] Stewart, Oscar M., *Physics, A Textbook for Colleges.* Ginn and Company, 1924.

This is the solution that is tabulated because this solution has been tabulated for the illustrative problem on page 451. Had the problem been given without any previous solution in which the above theorem in proportion was used, the solution would have been tabulated with the following steps:

Given: $\frac{I+i}{i} = 100$

$I + i = 100\,i$

$I = 99\,i.$

Formula: $\frac{I}{i} = \frac{R}{r}.$

Substituting: $\frac{99i}{i} = \frac{0.1}{r}$

$99r = 0.1$

$r = 0.00101.$

In the tables of this chapter, the column headed *Page* indicates a page of Stewart's text upon which a typical example of the case under consideration may be found. For teachers who may wish illustrative material, other pages upon which examples may be found are indicated in brackets after the statements. If there are problems on the page indicated, the number in parentheses indicates the number of the problem. The right-hand column indicates the frequency of occurrence in the text.

The geometry involved in freshman physics is quite elementary. The first 33 items of Table VIII are standard facts drilled upon in every class in plane geometry. The formula for the volume of a cylinder (item 34), the fact that surfaces of spheres are proportional to the squares of their diameters (item 36), and the fact that volumes of spheres are proportional to the cubes of their diameters (item 37) are usually known from the work in mensuration in arithmetic or have been presented informally in junior high-school mathematics. Item 35 is not connected with any fundamental principle in physics. It was used incidentally in the discussion of air currents, and probably does not occur in any other text. The facts indicated in items 38 and 39 are essential in understanding fundamental principles in physics, but are readily accepted without rigorous proof. The problem discussed at the first of this chapter illustrates the use of the theorem "If four quantities are in proportion, they are in proportion by composition." Problems of this type can be solved, as the illustration shows, without any reference whatever to this theorem.

TABLE VIII

GEOMETRICAL FACTS

GEOMETRICAL FACT	PAGE	FREQUENCY
1. The value of pi is 3.1416	128 (14)	44
2. The circumference of a circle equals $2\pi r$ [84, 111, 115, 123, 133, 188]	81	11
3. The sides of similar triangles are proportional [93, 95, 110, 114, 594, 615, 618, 644]	135	10
4. The square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides [14 (1, 2, 10, 12, 19) and 64 (1)]	64 (3)	7
5. Two triangles are similar if two angles of one equal two angles of the other [95, 110, 135, 615, 618]	664	7
6. The area of a circle equals πr^2 [35 (7), 188, 198, 538 (3, 4)]	197	7
7. Vertical angles are equal [110, 493 (5), 644]	618	5
8. Right angles are equal [95, 142, 644]	110	5
9. Alternate interior angles formed by a transversal cutting parallel lines are equal [378 (6), 406 (6), 594, 611]	609	5
10. Angles whose sides are perpendicular each to each are equal (if the proper angles are selected) [83, 628, 697]	693	4
11. The sum of the angles of a triangle is 180° [14 (11), 378 (6), 406 (6)]	493 (5)	4
12. The angles of an equilateral triangle are equal [594, 378 (6)]	406 (6)	3
13. The sides of an equiangular triangle are equal [406 (6)]	378 (6)	2
14. Corresponding angles formed by a transversal cutting parallel lines are equal [609]	135	2
15. A tangent to a circle is perpendicular to the radius at the point of tangency [611]	83	2
16. A right triangle having a 45° angle has two equal sides [14 (19)]	632	2
17. The area of a rectangle equals the product of the base and altitude [548 (6)]	594	2
18. Two ratios equal to the same ratio are equal to each other [618]	644	2
19. Two lines equal to the same line are equal to each other [611]	608	2

TABLE VIII (*Continued*)

Geometrical Fact	Page	Frequency
20. Two angles equal to the same angle are equal to each other	609	1
21. The sum of the angles about a point is 360°	14 (11)	1
22. The opposite sides of a parallelogram are equal	135	1
23. Perpendiculars to parallel lines are parallel	116	1
24. The areas of two circles are proportional to the squares of their diameters	188	1
25. Two right triangles with any two sides of one equal to the corresponding sides of the other are congruent	608	1
26. Two triangles having two angles and any side of one equal to the corresponding parts of the other are congruent	609	1
27. Corresponding angles of congruent triangles are equal	608	1
28. Corresponding sides of congruent triangles are equal	609	1
29. Complements of equal angles are equal	608	1
30. The sides opposite two equal angles of a triangle are equal	612	1
31. Parallel lines between parallel lines are equal.	644	1
32. A line may be substituted for its equal	644	1
33. If a point is on each of two lines, it is their point of intersection	614	1
34. The volume of a cylinder is $\pi r^2 h$	188	1
35. The surface of a sphere between two small circles, each parallel to a great circle, and 30° from it, is one half the total surface	305	1
36. The surfaces of two spheres are proportional to the squares of their diameters	126	1
37. The volumes of two spheres are proportional to the cubes of their diameters	126	1
38. The minimum surface inclosing a given volume is spherical	182	1
39. If a series of equidistant parallel lines cut a circumference, the lengths of the arcs between them decrease as the lines considered are nearer the center	514	1
40. If four quantities are in proportion, they are in proportion by composition [462 (17)]	451	2

From a study of the above list of geometrical facts, it would seem that we have no occasion to feel that the preparation of high-school graduates in this respect is inadequate.

The following tabulation of geometrical skills involves nothing but the simpler problems of geometry.

TABLE IX

GEOMETRICAL SKILLS

GEOMETRICAL SKILLS	PAGE	FREQUENCY
1. Construction of a line parallel to a given line..	613	2
2. Construction of a triangle with two given angles and the included side.....	5	2
3. Construction of a triangle with three given sides	6	1
4. Construction of a triangle with two given sides and the included angle.....	7	1
5. Drawing a figure to scale.....	622 (2)	19
6. Measuring and constructing angles by the use of a protractor.....	64 (2)	6
7. Diagrammatic representation of relationships...	15 (17)	59

The last three items in Table IX deserve special consideration. The standard texts in high-school geometry for the tenth grade do not systematically provide for practice in scale drawing. Many pupils graduate from high school without having had any experience with the protractor. Most courses in junior high-school mathematics make ample provision for scale drawing including the use of the protractor for making angles of a required size. They provide also for the use of the protractor in measuring given angles. As informal geometry is more generally taught, college freshmen will be better prepared for scale drawing and the use of the protractor. Ability to represent relationships by means of diagrams is not so much dependent upon the requirements of the course of study or the contents of the textbook as it is upon the attitude of the teacher. Any teacher who is convinced that diagrams are valuable in helping pupils to see relationships while solving problems, can easily train pupils in their use. If more work of the type advocated by Otis [2] were done in both arithmetic and algebra, the method of attack employed in these subjects might be carried over to physics, where it seems still more neces-

[2] Otis, Arthur S., "The Visual Method of Solving Arithmetic Problems," *The Mathematics Teacher*, Vol. XXI, No. 8, pp. 483–89 (December, 1928).

sary to have an actual picture of the relationships existing between the quantities and forces being studied.

In addition to the specific geometric facts listed in Table VIII, there are certain concepts in mathematics, of which some may be classified as geometric and some do not strictly belong in geometry. There are also certain facts that do not belong in Table VIII. Such facts and concepts appearing in Stewart's text are shown in Table X.

TABLE X

MISCELLANEOUS FACTS AND CONCEPTS

FACT OR CONCEPT	PAGE	FREQUENCY
1. Meaning of sin ϕ [628, 631, 693, 696]	626	11
2. Meaning of cos ϕ [95]	63	2
3. Meaning of tan ϕ	124	1
4. Sin ϕ is never greater than one	631	1
5. Sin $90° = 1$	632	1
6. If the denominator of a fraction is one unit of the measure being used, then the value of the fraction is expressed by the numerator	539	4
7. If L is a constant and I is a variable, rate of change $LI = L$ times rate of change of I	512	1
8. Multiplying the numerator and denominator of a fraction by 2 does not change its value	616	1
9. $(\sqrt{520{,}800})^2 = 520{,}800$	248 (4)	1
10. If $1/q = 0$, then q is infinitely large	616	1
11. If p is very large, then $1/p$ is approximately zero	616	1
12. For a small angle, an arc is approximately equal to its chord	83	2
13. Meaning and knowledge of percentage	430 (2)	11
14. Oppositeness of positive and negative [366, 383, 456, 532, 617, 620]	209	48
15. Projection of a point on a line	131	1
16. Parallels of Latitude on a sphere	147	1
17. Properties of a hinged parallelogram	116	2
18. Directed lines [83, 94, 135, 481]	61	24
19. Properties of Cavalierian bodies	29	1

From Table X, it is evident that the knowledge of trigonometry essential for freshman physics is not extensive. Many students who have not had trigonometry in high school do satisfactory

work in college physics. The general training in trigonometry is doubtless helpful, but so far as facts are concerned, the trigonometric facts needed in physics can be acquired in a short time.

Item 6 in Table X seems too elementary to mention if it merely means that $\frac{7}{1} = 7$, or that $\frac{a}{1} = a$. Pupils recognize this fact without difficulty. If we consider the formula $C = \frac{Q}{V}$, however, where $C =$ capacitance, $Q =$ charge, and $V =$ voltage, then the statement that "the capacitance of a condenser is equal to its charge when the charging voltage is equal to unity" is not so self-evident. It is the interpretation of such simple mathematical truths that the pupil is not able to make readily without previous experience with such interpretations.

The simple statement of item 7 in the language of physics will probably be accepted without any question. It is doubtful whether any general presentation of variables, constants, and limits in high school would improve the pupil's ability to grasp this concept.

The concepts expressed in items 10 and 11 are usually taught but are not stressed in high school. They occur so seldom in physics that increased emphasis in high school would be of doubtful value. Students will not question the truth of item 12 even though they have not heard of it in high school.

The different cases of percentage occur frequently in many college subjects. Algebra, geometry, and trigonometry as generally taught do not give many illustrations of percentage problems. "Percentage" to many students means problems dealing primarily with money situations. Percentage of error and percentage of efficiency do not belong in the same class with the percentage that they know. The questions of percentages in solutions as they arise in chemistry are not far removed from the types of problems that appear in algebra. The interpretation of Kelley's coefficient of alienation in the theory of statistics is an excellent illustration of the student's difficulty in thinking in the language of percentage. It is easily understood that

$$\frac{\sigma_{1\cdot 2}}{\sigma_1} = \frac{\sigma_1\sqrt{1-r^2}}{\sigma_1} = \sqrt{1-r^2}.$$

It is also evident that if $r = 0.80$, then $\sqrt{1-r^2} = 0.60$. Even knowing these facts, the student is surprised when he is told that this means that when $r = 0.80$, $\sigma_{1\cdot 2}$ is 60 per cent of σ_1.

The concept of oppositeness expressed by *positive* and *negative* was discussed in Chapter II. The two words actually occur more than two hundred times. The idea of opposite forces, opposite directions, and opposite characteristics of some sort appears in forty-eight cases. This fact further emphasizes the importance of so presenting the concept that pupils think of opposites when they consider positive and negative quantities.

The use of directed lines in physics frequently includes the addition of vectors. Although it is not advisable to confuse high-school pupils with the addition of vectors as found in the resultant of forces, yet it is perfectly possible to teach them to use directed lines.

Corresponding to Table X of miscellaneous facts and concepts, Table XI sets forth miscellaneous abilities found to be necessary for the mastery of Stewart's *Physics*.

Standard form numbers play an important part in the numerical problems of the sciences. Exceedingly large numbers and extremely small numbers occur so frequently that it is a great economy of time to represent them by standard form numbers. While most of the cases listed are merely the representation of large and small numbers, several of them show computations with standard form numbers. There are sufficient examples of each of the four operations to justify the demand for teaching addition, subtraction, multiplication, and division of expressions containing standard form numbers. It is true that pupils have had all the principles involved in these processes while studying the theory of exponents, but it is also true that unless they are given specific drill upon the actual type of computation required here, they will not be efficient in this work.

Students of science and of statistics have occasion to read values from various kinds of tables. The graduates of many high schools have had experience with only one kind of tables in their mathematics courses—logarithmic tables—and these are not the tables which they are obliged to use in beginning college courses in physics, chemistry, or statistics. It is often convenient to be able to use tables of logarithms, but they are not an absolute necessity. Tables of squares, of square roots, and of natural trigonometric functions are used much more frequently during the first year. Linear interpolation is also necessary. To many students, interpolation is a process belonging exclusively to

TABLE XI

MISCELLANEOUS ABILITIES

ABILITY	PAGE	FREQUENCY
1. Expressing large numbers as standard form numbers	88 (17)	35
2. Multiplying a standard form number by an ordinary number [179, 197, 226 (7), 264 (8)]	248 (11)	8
3. Multiplying a standard form number by a standard form number [202 (9)]	179	2
4. Adding a standard form number to a standard form number	35 (7)	2
5. Dividing an ordinary number by a standard form number [161 (7), 198, 202 (5, 6, 8)]	107 (3)	10
6. Dividing a standard form number by a standard form number	179	1
7. Expressing small numbers in the form of standard form numbers	697	12
8. Reading a value from a table without interpolation	107 (3)	10
9. Reading a value from a table using interpolation	316 (5)	1
10. Finding the square root of a number (any method)	64 (3)	15
11. Finding the sine of an angle from a table	627	3
12. Finding the cosine of an angle from a table	63	2
13. Representing a relationship between two variables by means of a line graph [475, 508, 517, 528, 583]	218	16
14. Interpreting a line graph [528, 529]	527	4
15. Graphically representing the combining of two functions of one variable by adding the ordinates of the curves	332	2
16. Shifting from geometric to algebraic representation of relationship	348	2

logarithmic computation. If pupils graduate from high school without having had practice in reading values from tables and in interpolation the school has failed to prepare them in a simple and important process.

The representation of relationships by means of graphs is found

in physics, chemistry, and statistics. In fact, it is found in so many fields that the teachers of high-school mathematics and authors of high-school texts, recognizing the demand for this ability, are in general, providing adequate preparation in it. The ability to interpret graphs and to "read off" values from a graph is essential in both physics and chemistry. Pupils are not being so well trained in this as in drawing graphs, but this phase of the work is receiving more attention now than formerly.

It is not likely that pupils have had much experience in adding ordinates in graphic work, unless they have had it in connection with graphs of trigonometric equations. There is not sufficient need of it in physics to justify any special effort to teach it.

The transitions from geometrical to algebraic expressions of relationship are not at all difficult. Most pupils are getting more of this than is used in physics. The tendency to offer more of the simple elements of analytic geometry in the high-school courses will increase this valuable training.

The various types of algebraic equations in one variable which arise from numerical substitution in physics formulas are listed in Table XII. Each equation given is an actual case from the text, but frequently a different case from that indicated in the *page column*. The frequency for each case has been obtained by counting all equations of the same algebraic form, differing only in the numbers or the unknowns, or both.

Table XII shows that the algebra taught in high school is quite different from the algebra required in physics. Parentheses occur in Table XII but there are no cases of parentheses within parentheses. Nor are there instances of complicated expressions within the parentheses, or of products of parenthetical expressions of two or more terms by other parenthetical expressions of two or more terms. The fractional equations are rather simple. A few of them have a binomial in either the numerator or the denominator. There are no cases in which two binomials occur in one equation. No equation has more than two terms in any numerator or denominator. All equations having two or more denominators have monomial denominators. This means that in the process of clearing of fractions, the only *least common multiple of the denominators* that needs to be found is that of one-termed expressions. The only instances of radical equations have one radical by itself on one side of the equation, and one term only on the other side, so

TABLE XII

Algebraic Equations in One Variable

Type of Equation	Page	Frequency
1. $4F = 100$	118 (1)	14
2. $500 = 1000a$	638 (6)	16
3. $5280 = a(20)$	64 (9)	22
4. $5x + 40 = 150$	15 (14)	5
5. $5775.52 = 78.57 + p_3$	175 (2)	4
6. $4785.3 = 50.9H + (50.9)(13.5)$	269	3
7. $30x + 50(x - 6) - 60(12 - x) = 0$	13	5
8. $200(t - 10) = 500(30 - t)$	234 (3)	3
9. $50(0.056)(t - 16) + 200(t - 16) = 65(0.056)(100 - t)$	234 (6)	3
10. $400(40 - t) = 25(79.8) + 25t$	291 (1)	2
11. $5390 + 10(100 - t) = 50(79.8) + 50t + 400(t - 30) + 200(.09)(t - 30)$	292 (13)	2
12. $20 = 2.2 + 5(0.1 + R)$	460 (22)	1
13. $3 = 0.011(5 + x)$	494 (9)	2
14. $250(79.8) + 250t = 400(60 - t) + 18(60 - t)$	292 (7)	1
15. $75.6 = \frac{F}{16}$	638 (3)	9
16. $5 = \frac{15}{32}a$	74	13
17. $5500 = F\frac{88}{3}$	108 (20)	1
18. $8.4 = \frac{1000}{v}$	52 (13)	3
19. $0.8 = \frac{25}{25 - x}$	36 (15)	1
20. $8.9 = \frac{x}{x - 100}$	35 (12)	1
21. $100 = \frac{0.1 + R}{R}$	462 (17)	1
22. $\frac{x}{16} = 6$	605 (3)	16
23. $\frac{1}{p} = 1$	652 (16)	1
24. $\frac{0.001118}{v} = \frac{167.88}{8}$	476 (6)	4
25. $\frac{1}{R} = \frac{7}{20}$	448	8
26. $\frac{AB}{200} = \frac{6}{10}$	93	9
27. $\frac{73v}{76(774)} = \frac{293}{273}$	227 (14)	2

TABLE XII (*Continued*)

Type of Equation	Page	Frequency
28. $\frac{1}{24} = \frac{1}{f}$	652 (10)	6
29. $\frac{98.8}{86.2} = \frac{t+273}{0+273}$	227 (13)	1
30. $\frac{24}{14{,}400} = \frac{x}{6{,}400}$	605 (5)	2
31. $\frac{1}{10} = \frac{2}{R}$	622 (8)	1
32. $\frac{1}{60} + \frac{1}{q} = \frac{2}{30}$	622(1)	18
33. $\frac{1}{p} + \frac{1}{240} = \frac{1}{8}$	644 (7)	4
34. $1\frac{1}{2}x = 12$	118 (2)	1
35. $1000 = \frac{1}{6.2832\sqrt{0.000001\,L}}$	591 (2)	1
36. $1 = 2\pi\sqrt{\frac{l}{980}}$	138 (10)	2
37. $1.9 = 2\pi\sqrt{\frac{90}{g}}$	139 (9)	1
38. $(0.000012)(100)t = (0.000018)lt$ (Solve for l)	226 (5)	2
Quadratic Equations		
39. $557 = 16t^2$	65 (14)	6
40. $0.97v^2 = 5760$	175 (5)	2
41. $64 = \frac{m^2}{25}$	378 (2)	1
42. $\frac{16}{9} = \frac{300}{d^2}$	605 (2)	1
43. $2 = \frac{4(9.86965)\,5/32}{T^2} \cdot \frac{1}{12}$	137 (3)	2

that squaring both sides of the equation removes all radicals without any opportunity for the errors that so often occur in algebra when there are two or more terms on one side of the equation.

The only quadratic equations that occur are of the form $557 = 16t^2$. Not a single case of a general quadratic equation appears in Stewart's text. Spinney's [3] text has one problem stated as

[3] Spinney, Louis Bevier, *A Textbook of Physics*, p. 32. The Macmillan Company, 1925.

follows: "A ball is thrown from the top of a tower 200 ft. high with a velocity of 30 ft./sec. What will be the time of flight if thrown vertically upward? If thrown horizontally? If thrown downward?" Two similar problems are found in Smith's [4] text. These three problems give rise to a general quadratic equation in t when the numerical values for u, s, and g are substituted in the formula $s = ut + 1/2\, gt^2$. It should be noted that the resulting equations are not factorable. There are two roots, irrational and opposite in sign, for each of the three equations. The student must recognize the reason for discarding the negative root in each case. It is possible that other problems in freshman physics texts give rise to general quadratic equations, but they are not frequent. It is well known that they are numerous in advanced courses in physics and mechanics. The Kilzer-Kirby Inventory Test mentioned in Chapter I has three cases of the pure quadratic, but none of the general quadratic, from which we are led to infer that Dr. Kilzer did not find any general quadratic equations in his examination of the five high-school texts.

Equations with two variables furnish another excellent illustration of the difference of emphasis in algebra and in physics. Every high-school text in algebra has the ordinary simultaneous equations of the form

$$\begin{aligned} 3x + 2y &= 4 \\ 5x - \ y &= 11. \end{aligned}$$

These equations are solved by *addition and subtraction,* by *comparison,* and by *substitution* in many of the books, and then a graphic solution is always discussed. No problems of this type appear in either the expository material or the problems in Stewart's text. The problems that do occur are of the form

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{1}{5} \\ \frac{p}{q} &= \frac{1}{6} \end{aligned}$$

or the form

$$\begin{aligned} p + q &= 15 \\ \frac{p}{q} &= \frac{1}{2}. \end{aligned}$$

[4] Smith, Alpheus, *The Elements of Physics,* p. 11. McGraw-Hill Book Company, 1927.

The nearest approach to this type of problem that is generally found in algebra textbooks is the following:

$$\frac{6}{x}+\frac{12}{y}=-1$$
$$\frac{8}{x}-\frac{9}{y}=\quad 7.$$

Such problems are not numerous in algebra. They are generally considered as unusual or exceptional problems, or even as *catch problems* in which the trick is to eliminate *without clearing of fractions.* The physics problems illustrated above are not quite in this form. On the contrary, it is necessary to *clear of fractions;* or at any rate it is more convenient to clear of fractions in one of the two equations. Ligda [5] has given one plausible argument for continued emphasis upon the ordinary simultaneous equations of the first degree with two unknowns. He says that an experiment with two classes showed that the class in which simultaneous equations were introduced very early did superior work especially with parentheses. Simultaneous equations are helpful also in teaching the idea of locus. Determining the constants in simultaneous equations for given values of x and y probably has more applications in college subjects than does the ordinary solution of such equations.

The types of algebraic equations in one unknown which are set forth in Table XII involve the major portion of algebraic ability required for solution of problems in physics. A careful examination of this table reveals no problems that are intrinsically difficult. None of the difficult types of algebraic equations involving complicated parentheses, complicated fractions, or complicated radicals appears; and yet these equations are troublesome. Although they are not complicated, they are unusual. The numbers are large and awkward. There are many more decimals than pupils find in high-school algebra. A mixture of decimals and common fractions frequently occurs in the same problem, such as those in numbers 24, 29, and 43. This is unusual in high-school algebra. The letters are different. Solving for F, p_2, H, t, ϕ, and R presents a slightly different situation from that presented in solving for x.

The manipulations of formulas are shown in Tables XIII, XIV,

[5] Ligda, Paul, *The Teaching of Elementary Algebra,* p. 64. Houghton Mifflin Company, 1925.

and XV. Table XIII is a list of algebraic manipulations in which the literal value of a letter is substituted in an equation in order to eliminate that letter. In Table XIV, the expression of a relationship between two or more variables is changed into a more desirable form. Table XV is a miscellaneous list of algebraic manipulations.

TABLE XIII

LITERAL SUBSTITUTIONS

VALUE SUBSTITUTED	EXPRESSION IN WHICH IT IS SUBSTITUTED	PAGE
1. $v = u + at$	$\bar{v} = 1/2\,(u + v)$	57
2. $a = \frac{v_2 - v_1}{t}$	$F = ma$	77
3. $\bar{v} = u + 1/2(at)$	$s = \bar{v}t$	57
4. $a = \frac{v^2}{r}$	$F = ma$	84
5. $v = 2\pi rn$	$F = \frac{mv^2}{r}$	84
6. $AD = AC \cos \theta$	Work $= AB \cdot AD$	95
7. $s = 1/2(at^2)$	$W = mas$	99
8. $F = ma$	$W = Fs$	99
9. $a = \frac{v}{t}$	$W = 1/2\,(ma^2t^2)$	99
10. $\frac{d}{s} = \frac{a}{b}$	$F = \frac{d}{s}w$	110
11. $F = fF'$	Power $= Fv$	124
12. $a = \frac{v^2}{r}$	$a' = a\frac{x}{r}$	132
13. $s = 2\pi r$	$s = vt$	133
14. $\frac{v}{r} = \frac{2\pi}{T}$	Force $= m\frac{v^2}{r^2}x$	133
15. $F = \frac{4\pi^2 m}{T^2}x$	$F = \frac{mg}{l}x$	135
16. $l = \frac{I}{mR}$	$T = 2\pi\sqrt{\frac{l}{g}}$	136
17. $v_1 = \omega r_1$ $v_2 = \omega r_2$ etc.	$K.E. = 1/2(m_1v_1^2) + 1/2(m_2v_2^2)$ etc.	149
18. $\omega = 2\pi N$	$K.E. = 1/2\,(I\omega^2)$	149
19. $\frac{v_t}{v_o} = 1 + \beta t$	$\frac{h_t}{h_o} = \frac{v_t}{v_o}$	218
20. $t = T - 273$ and $p_ov_o = 273R$	$pv = \frac{p_ov_o}{273}(273 + t)$	222

TABLE XIII (*Continued*)

Value Substituted	Expression in Which It Is Substituted	Page
21. $T = \frac{1}{n}$	$V = \frac{\lambda}{T}$	331
22. $\lambda' = \frac{V - v}{n}$	$n' = \frac{V}{\lambda'}$	349
23. $M = ml$	Torque $= mHl$	376
24. $Q = It$	Work $= Q \cdot$ Voltage	428
25. $V = Ir$	$W = IVt$	439
26. $E = Ne$ and $R' = R + Nr$	$I = \frac{E}{R'}$	454
27. $R' = R + \frac{r}{N}$ and $E = e$	$I = \frac{E}{R'}$	455
28. $N = \frac{Q}{ev}$	$M = Nm$	470
29. $I = kd$	$V = Ir$	489
30. Ave. $E.M.F. = N_2 \frac{\phi}{t}$	Ave. Current $= \frac{\text{Ave. } E.M.F.}{R_2}$	524
31. Ave. Current $= N_2 \frac{\phi}{R_2 t}$	$Q =$ Ave. Current $\cdot t$	524

The manipulations of literal formulas presented in Tables XIII, XIV, and XV do not occur in the solutions of problems, but are necessary in the development of formulas used in these problems. Although many of them appear but once, it is necessary that the student know how to use them this one time in order to develop a formula that is used many times.

In one sense, the illustrations in Table XIII may be considered *simultaneous equations* in which an unknown is eliminated by substitution. It is doubtful, however, whether practice in elimination by substitution in ordinary simultaneous equations will in any manner make a student more efficient in performing these substitutions. The same thing can be said concerning this list that was said about the items in Table XII, "They are not intrinsically difficult, but unusual." If each of the 31 items in Table XIII is carefully examined, it is found that the mechanical substitution of the given value in the expression indicated and the simplification of the resulting expression involve nothing but fundamental

TABLE XIV

Changing the Form of an Equation

Given Equation	New Form	Page
1. $Fb = Wa$	$F = \frac{a}{b} W$	10
2. $0.433h_2d = 0.433h_1$	$d = \frac{h_1}{h_2}$	33
3. $a = \frac{v'}{t}$	$v' = at$	83
4. $\frac{vt}{r} = \frac{at}{v}$	$a = \frac{v^2}{r}$	84
5. $FR = \frac{1}{2}w(R-r)$	$F = \frac{w}{2R}(R-r)$	114
6. $F = \frac{w}{2R}(R-r)$	$\frac{w}{F} = \frac{2R}{R-r}$	114
7. $F = \frac{d}{s}w$	$\frac{w}{F} = \frac{s}{d}$	114
8. $2\pi rF = wd$	$F = \frac{d}{2\pi r}w$	115
9. $\frac{a'}{a} = \frac{x}{r}$	$a' = a\frac{x}{r}$	132
10. $2\pi r = vT$	$T = \frac{2\pi r}{v}$	133
11. $T = \frac{2\pi r}{v}$	$\frac{v}{r} = \frac{2\pi}{T}$	133
12. $\frac{F}{mg} = \frac{x}{l}$	$F = \frac{mg}{l}x$	135
13. $\frac{mg}{l}x = \frac{4\pi^2 m}{T^2}x$	$T^2 = 4\pi^2\frac{l}{g}$	135
14. $T^2 = 4\pi^2\frac{l}{g}$	$T = 2\pi\sqrt{\frac{l}{g}}$	135
15. $G\frac{Mm'}{R^2} = m'g$	$GM = gR^2$	157
16. $\frac{1}{2}\frac{w}{g}v^2 = wh$	$v = \sqrt{2gh}$	172
17. $2\pi rS = h\pi r^2 dg$	$S = \frac{hrdg}{2}, \quad h = \frac{2S}{rdg}$	188
18. $M = \frac{Fl}{eA}$	$F = M\frac{eA}{l}, \; e = \frac{Fl}{AM}, \; A = \frac{Fl}{eM}$	197
19. $l_t - l_o = \alpha l_o t$	$l_t = l_o(1+\alpha t)$	212
20. $h_o d_o g = h_t d_t g$	$\frac{d_o}{d_t} = \frac{h_t}{h_o}$	217
21. $v_t = v_o(1+\beta t)$	$\frac{v_t}{v_o} = 1+\beta t$	218

TABLE XIV (*Continued*)

Given Equation	New Form	Page
22. $V = n'\lambda'$	$n' = \frac{V}{\lambda'}$	349
23. $i_1 r_1 = i_2 r_2$	$\frac{i_1}{i_2} = \frac{r_2}{r_1}$	450
24. $\frac{i_1}{I} = \frac{r_2}{r_1 + r_2}$	$i_1 = I \frac{r_2}{r_1 + r_2}$	451
25. $IE = I^2R + Ie$	$I = \frac{E - e}{R}$	459
26. $Q = Nev$	$N = \frac{Q}{ev}$	470
27. $Q = N_2 \frac{\phi}{R_2}$	$\phi = Q \frac{R_2}{N_2}$	524
28. $\frac{R - p}{q - R} = \frac{p}{q}$	$\frac{1}{p} + \frac{1}{q} = \frac{2}{R}$	618
29. $\frac{\sin i_1}{\sin i_2} = \frac{4}{3}$	$\sin i_1 = 4/3 \sin i_2$	631
30. $1 = n \sin i_2$	$\sin i_2 = \frac{1}{n}$	632
31. $\frac{\sin i_1}{\sin i_2} = \frac{n_2}{n_1}$	$n_1 \sin i_1 = n_2 \sin i_2$	634
32. $\frac{p}{q} = \frac{f}{q - f}$	$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$	645
33. $\frac{1}{p} - \frac{1}{q} = \frac{1}{f}$	$\frac{q}{p} = \frac{q}{f} + 1$	657
34. $\sin \phi = \frac{AB}{d}$	$AB = d \sin \phi$	693
35. $\frac{x}{r_1} = \frac{r_3}{r_2}$	$x = \frac{r_1}{r_2} r_3$	446
36. $\frac{1}{R} = \frac{1}{r} + \frac{1}{r} + \frac{1}{r}$	$R = \frac{r}{3}$	449
37. $F = m \frac{v_2 - v_1}{t}$	$Ft = mv_2 - mv_1$	77
38. $\frac{AB}{AC} = \frac{AE}{AD}$	$AB \cdot AD = AC \cdot AE$	95
39. $IR = Ir_1 + Ir_2 + Ir_3$	$R = r_1 + r_2 + r_3$	448
40. $K = \frac{pv}{\text{change in volume}}$	$\frac{p}{K} = \frac{\text{change in volume}}{v}$	202 (10)
41. Energy = Weight $(l_2 - l_1)$	$l_2 - l_1 = \frac{\text{Energy}}{\text{Weight}}$	426
42. Volts = amperes × ohms	$\text{amperes} = \frac{\text{volts}}{\text{ohms}}$	422
43. Work = $Q \times E.M.F.$	$\frac{\text{Work}}{t} = \frac{Q}{t} \times E.M.F.$	431

principles that have been taught in all algebra classes. Probably most teachers will say that their pupils have been trained in the fundamental steps into which these problems can be analyzed, and that therefore they will have no serious difficulty in making the substitutions and in simplifying the results. The facts are that although the pupils have had experience with the elements involved in the simplifications of these expressions, they have very seldom, if ever, met these particular and peculiar combinations of simple elements in a situation so complex as many of these. Certainly, they have not met them with this variety of capital letters, small letters, Greek letters, primes, and subscripts.

The first 39 items of Table XIV, page 47, show another type of algebraic manipulation. An expression is given in a certain form, and then is transformed into a more desirable equivalent form. Many of these conversions consist in *changing the subject* [6] by merely dividing or multiplying both sides of the equation by some letter or letters. This is a simple operation in which the pupil has received ample drill. Solving for a ratio between two letters as in 6, 7, 11, 20, 21, and 23 is a type of problem which is rare in the ordinary text. It is not difficult, merely different. Number 6, for example, does not seem especially difficult. We shall see in Chapter VI, nevertheless, that only 53.6 per cent of those who tried to solve it were able to make the correct transformation. Extracting the square root of both sides of an equation is a well-known step, but it does not usually occur in connection with such little *tricks* as those illustrated in numbers 14 and 16. The kind of factoring which occurs in number 19 is representative of the difficulty of factoring in physics problems. In number 25 the fact that I occurs in every term presents a situation that is confusing. This problem is discussed in Chapter VI. Number 27, which does not seem complicated, has in it a new element of confusion, which is discussed in Chapter VI. The peculiar transformations in numbers 28, 32, and 33 are really the only ones in the entire table that appear difficult. In these three cases, the text outlines the intermediate steps to be taken. Number 36 should be mentioned as one that is so unusual and yet so simple that the answer is likely to come as a surprise to the student. The four equations at the end of Table XIV are merely illustrative of the

[6] Solving for a different letter, see Nunn, T. Percy, *The Teaching of Algebra (Including Trigonometry)*, pp. 77, 78. Longmans, Green & Company, 1927.

various types of manipulations applied to rhetorical equations. There are one hundred four of these equations in Stewart's text, but probably they do not offer any new difficulties further than the fact that since these particular situations have not been met in algebra, the student may be unable to make the immediate transfer from numerical to rhetorical transformations.

In Table XV numbers 1 and 2 indicate the maximum of difficulty of expressions with parentheses in literal quantities. The

TABLE XV

MISCELLANEOUS ALGEBRAIC MANIPULATIONS

ORIGINAL FORM	CHANGED FORM	PAGE
1. $1/2\ (u+u+at)$	$u+1/2\ (at)$	57
2. $(u+1/2\,at)t$	$ut+1/2\,at^2$	57
3. $1/2\ wR-1/2\ wr$	$1/2\,w(R-r)$	113
4. $F_1=Ap_1$ and $F_2=Ap_2$	$F_1-F_2=A(p_1-p_2)$	29
5. $Fx+Fy$	$F(x+y)$	13
6. $\frac{F\cdot 2\pi r}{2\pi R}$	$\frac{r}{R}F$	123
7. $1/2\,m_1\omega^2r_1^2+1/2\,m_2\omega^2r_2^2$ etc.	$1/2\,\omega^2(m_1r_1^2+m_2r_2^2)$ etc.	149
8. $\frac{F/A}{e/l}$	$\frac{Fl}{eA}$	196
9. $p_0v_0(1+\frac{1}{273}t)$	$\frac{p_0v_0}{273}(273+t)$	222
10. $p_1v_1=RT_1$ and $p_2v_2=RT_2$	$\frac{p_2v_2}{p_1v_1}=\frac{T_2}{T_1}$	222
11. $i_1x=i_2r_2$ and $i_1r_1=i_2r_2$	$\frac{x}{r_1}=\frac{r_2}{r_2}$	447
12. $I=V\frac{1}{R}$ and $I=V\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$	$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$	450
13. $M=\epsilon\ Q$ and $M=\frac{m}{ev}Q$	$\epsilon=\frac{m}{ev}$	470
14. $\frac{V}{r_1}+\frac{V}{r_2}+\frac{V}{r_3}$	$V\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$	449
15. $Ir_1+Ir_2+Ir_3$	$I(r_1+r_2+r_3)$	453
16. $Q_1=C_1V, Q_2=C_2V, Q_3=C_3V$	$Q_1+Q_2+Q_3=(C_1+C_2+C_3)V$	543
17. $Q_1+Q_2+Q_3=CV$ and $Q_1+Q_2+Q_3=(C_1+C_2+C_3)V$	$C=C_1+C_2+C_3$	543
18. $I=CEv$ and $E=LIv$	$v=\frac{1}{\sqrt{CL}}$	588

TABLE XV (*Continued*)

Original Form	Changed Form	Page
19. $Q_1=C_1V_1, Q_2=C_2V_2, Q_3=C_3V_3$ and $V=V_1+V_2+V_3$	$V=\frac{Q_1}{C_1}+\frac{Q_2}{C_2}+\frac{Q_3}{C_3}$	544
20. $\frac{ab}{a'b'}=\frac{r_1}{r_2}$ and $\frac{bc}{b'c'}=\frac{r_1}{r_2}$	$\frac{ab\cdot bc}{a'b'\cdot b'c'}=\frac{r_1^2}{r_2^2}$	594
21. $\sin i_1=\frac{BC}{AC}$ and $\sin i_2=\frac{AD}{AC}$	$\frac{\sin i_1}{\sin i_2}=\frac{BC}{AD}$	628
22. $n_1=\frac{V_0}{V_1}, n_2=\frac{V_0}{V_2}$	$\frac{n_2}{n_1}=\frac{V_1}{V_2}$	634

most difficult factoring found in Stewart's text is represented in numbers 3, 4, 5, 7, 14, 15, and 16; in other words, the taking out of a monomial factor, either fractional or integral, is the only kind of factoring that occurs. It should be mentioned that taking out a common monomial factor which is fractional probably does not occur one-tenth as often in high-school algebra as three or four other types of factoring which do not appear a single time in this physics text. The most difficult case of simplification of complex fractions is number 8, in which both numerator and denominator are fractions made up of a single term in the numerator and a single term in the denominator. The combination of two equations into one by the division of member by member is illustrated in numbers 10, 11, 21, and 22. The combination of equations by the multiplication of member by member is found in number 20. The addition or the subtraction of equations, member by member, occurs in 4 and 16.

It is not likely that any one of the operations in Table XV mentioned so far needs more emphasis than is now given to it in algebra. The transformation represented in number 9 is not so common. Probably the student has never been called upon to do anything of the sort. The elimination of I and V in number 12 and a similar trick in number 17 are readily understood even though such a process is seldom found in high-school texts.

Numbers 13, 18, and 19 are also illustrations of elimination of one or more letters by unconventional methods. It may be contended that there is not a single step in numbers 9, 12, 13, 17, 18, and 19 in which high-school pupils are not thoroughly drilled. Although this be true, the pupils may not be able to carry through to completion the steps as they are combined in these problems.

As a partial check upon the findings in Stewart's text, the first 69 pages of Kimball's *A College Textbook of Physics* were studied, and the first 52 problems of this text were solved. The mathematics involved differs slightly from that in Stewart's text. There are applications of two additional geometry theorems: "The area of a triangle equals one-half the product of the base by the altitude," and "Two angles whose sides are respectively parallel are either equal or supplementary." The law of sines and the law of cosines are mentioned once each. The subjects discussed do not come in the same order, but for corresponding subjects the same formulas with slight variations appear in the two books. One book uses v_1 and v_2 for initial and final velocities, whereas the other uses u and v. One book makes use of the double colon in the expression of a proportion, and the other has the equality sign. Solving the proportion $F : Mg :: h : l$ for F illustrates the most noticeable difference between the actual algebraic manipulations in the two texts. The pure quadratic equation appears five times. There are no cases of the general quadratic. The taking out of the common monomial factor, $W/2$, occurs once. Nothing more difficult in factoring appears. Clearing of fractions with monomial denominators occurs, but there are no cases with more than one term in the denominator. Parentheses within parentheses do not occur; nor do parentheses with more than two terms.

The actual amount of mathematical knowledge and skill required in freshman chemistry is much less than that required in freshman physics. Many of the arithmetic computations are complicated, but the algebra is very simple. The subject of molecular concentration (or mass action) is presented algebraically by the four texts A, B, C, and E of those listed in Chapter III. The following statements are typical, the only difference being in the choice of symbols:

$$k[PCl_5] = k'[PCl_3][Cl_2],$$

Whence

$$\frac{[PCl_3][Cl_2]}{[PCl_5]} = \frac{k}{k'} = K.$$

A complete understanding of these equations probably involves the most difficult algebra found in the texts. There are five distinct difficulties:

1. The algebraic statement of the law that the speed of the action is proportional to (or varies as) the molecular concentration.

$$k[PCl_5] = s \qquad \text{also} \qquad k'[PCl_3][Cl_2] = s'$$

The chemical symbol $[PCl_5]$ used to represent the molecular concentration of phosphorus pentachloride instead of a single letter x as in algebra offers one new situation. The expression of *varies as* by means of putting the variable s equal to a constant k times another variable has not appeared before in any of the texts. One of them uses the infinity sign for *varies as,* and then changes to the equation form with the constant.

2. For a system in chemical equilibrium, the speeds of the forward and reverse reactions must be equal.

$$k[PCl_5] = k'[PCl_3][Cl_2]$$

The chief difficulty here is in the use of these symbols in place of the single letter for a variable.

3. The solution of this equation for the ratio $\frac{[PCl_3][Cl_2]}{[PCl_5]}$.

Even if these were simple algebraic symbols, this step would offer some difficulty since students have usually solved for a single letter rather than for a ratio.

4. The substitution of a new constant K for the fraction k/k'. This is a procedure seldom met in high school.

5. The practical interpretation of the resulting formula to mean that after the *equilibrium constant* K has been determined, then any change in the molecular concentration of any one of the three variables necessitates a change in one or both of the remaining variables. It requires considerable experience with fractions to appreciate the variety of changes that might keep the value of the fraction constant when some one variable either increases or decreases. The equations in some of the texts involve two variables in the numerator and two in the denominator.

Two of the texts (A and B) extend this symbolism to express the law $[H_2] \times [H_2] \times [O_2] \times k =$ speed, in the form $[H_2]^2 \times [O_2] \times k =$ speed. One of them uses exponents 3 and 4 in this manner. These are the only instances in the five texts where a variable appears with an exponent of 2 or more.

Texts A, B, and D employ the algebraic form to express the two laws of gases:

$$\frac{P_1}{P_2} = \frac{V_2}{V_1} \quad \text{and} \quad \frac{V_1}{V_2} = \frac{T_1}{T_2}.$$

In Text A, each of these formulas is solved for V_1. In text B, the two formulas are combined into the one formula,

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}.$$

This same text has two other formulas in connection with volumes which involve more algebraic manipulation than is found in the other texts:

$$\frac{760 \times V_s}{273} = \frac{PV}{T}, \quad \text{whence } V_s = \frac{P \times V \times 273}{760 \times T} \quad \text{and}$$

$$V_s = \frac{(P - a) \times V \times 273}{760 \times T}.$$

Although text D does not present molecular concentration algebraically, it utilizes algebra more than any of the others in stating other laws. $W = Mg$ and $D = M/V$ are two of the laws so stated. Two rhetorical equations are used. Transposition is employed in the first one thus: weight of copper + weight of oxygen = weight of copper oxide, whence, weight of copper oxide − weight of copper = weight of oxygen. The other merely states a law:

$$\frac{\text{Atomic Weight}}{\text{Valence}} = \text{Equivalent Weight.}$$

This text also gives the standard formulas for transforming temperatures from Fahrenheit to centigrade, and vice versa.

The ordinary form of a proportion occurs frequently in texts A, B, and D, but in every case there are three numerical terms with the one unknown. Problems requiring the solution of such a proportion occur in all of the texts more frequently than any other one type of problem.

All the texts have numerous tables, such as that for atomic weights, which must be used in many of the problems.

Line graphs showing the relation between two variables occur in all the texts. Solubility curves showing the relationship between temperature and solubility for various salts appear in each of the five texts. These are sufficiently accurate for the student to *read off* the solubility for a given temperature, or vice versa. Curves showing the relationship between temperature and vapor pressure appear in texts B, D, and E. Text A shows the growth of the nitrogen fixation industry by means of a graph, and text B shows the petroleum output for a series of years by means of a graph.

The actual facts from Euclidean geometry are not used in any of the five texts. Familiarity with geometric forms, however, is essential in the study of crystals. In order to understand the descriptions occurring in texts A, B, and D, the student must definitely understand the following expressions: three equal axes of symmetry at right angles; two equal axes of symmetry, and one of different length, all at right angles; three unequal axes of symmetry at right angles; two axes of symmetry at right angles and a third at a different angle; three axes of symmetry of any length and at any angle of inclination; three equal axes of symmetry in the same plane intersecting at angles of sixty degrees, and a fourth at right angles to all of these.

Geometrical figures are used in each of the five texts to represent diagrammatically such laws or theories as that of atomic structure.

No reference to trigonometry occurs in any of these five chemistry textbooks.

A study of the tabulations of this chapter may easily lead to the wrong inference. Negative evidence is never convincing. The facts, concepts, and skills that occur should certainly be taught somewhere. The inference that those not found in this investigation need not be taught is likely to lead to difficulties inasmuch as definite use for them may arise elsewhere. Furthermore, drill in manipulations which may seem of little intrinsic value may be essential for the comprehension of that which follows. Computation is not an end in itself. Judd [7] says, "We pass pupils through

[7] Judd, C. H., "Informational Mathematics Versus Computational Mathematics," *The Mathematics Teacher*, Vol. XXII, No. 4, p. 194 (April, 1929).

the computational stage for the purpose of bringing them to the higher levels of information."

A comparison of the findings of this chapter with Tables 1 and 2 of the Appendix shows quite general agreement. If frequency were accepted as the criterion for being essential, the order of the topics listed in Table 2 would be somewhat changed. Variation would come much higher in the list and quadratic equations in one unknown and simultaneous linear equations in two unknowns would be in about the middle of the list. It seems rather misleading, however, to attempt to determine an absolute order of importance by either the questionnaire method or the frequency of occurrence. In Table 1 of the Appendix, variation would probably be changed from group 3 to group 1, and numerical trigonometry would probably be changed from group 3 to group 2 if frequency of occurrence were the deciding factor. Probably the greatest discrepancy is revealed by a comparison with Thorndike's [8] report. He says that every one of seventeen teachers of college physics reported as *essential* the solution of the pair of equations

$$2x + 5y = 15$$
$$3x - 4y = 11.$$

No problem of this type occurs in Stewart's text. On the other hand, two problems in standard form numbers were scored 2.6 and 2.5 on a scale in which these simultaneous equations were scored 3. Standard form numbers occurred seventy times in Stewart's text.

[8] Thorndike, E. L. and others, *The Psychology of Algebra*, pp. 59–66. The Macmillan Company, 1923.

CHAPTER V

GENERAL PROCESSES AND METHODS OF PROCEDURE

It is a comparatively simple task to tabulate words, symbols, facts, and specific skills which are involved in the study of college physics. These are important, but still more important are those mental activities which the National Committee Report [1] designates as *habits of thought:*

> The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual. . . . Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes, and principles in the solution of concrete problems rather than on the acquisition of mere facility or skill in manipulation. The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. On the side of algebra, the ability to understand its language and to use it intelligently, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims.

The ease with which tests of the manipulative skills can be devised and administered as compared with the difficulties in testing powers of analysis, generalization, interpretation, and grasp of underlying principles has not tended to improve the conditions referred to above as "the main obstacles to intelligent progress."

Hedrick [2] has suggested that perhaps the truest values in mathematical training lie in training in processes rather than in facts and specific skills, since the transfer of facts and skills is far more problematic than the transfer of general processes. Everett [3] ex-

[1] National Committee Report, pp. 10, 11.

[2] Hedrick, E. R., "The Reality of Mathematical Processes," *Third Yearbook of the National Council of the Teachers of Mathematics,* pp. 35–41. Bureau of Publications, Teachers College, Columbia University, 1928.

[3] Everett, J. P., *The Fundamental Skills of Algebra,* p. 3. Contributions to Education, No. 324. Bureau of Publications, Teachers College, Columbia University, 1928.

presses the same idea in the statement "the child needs algebra for what it will enable him to do, but infinitely more does he need it for the orderly way in which, in the midst of constant change, it will enable him to think."

It is admitted that a list and evaluation of processes and procedures must be much more subjective than a list of words, symbols, facts, or skills. In the light of the above opinions it would seem that even an approximately accurate list of such general processes and methods of procedure would prove to be exceedingly valuable. This chapter is devoted to the discussion of such a list.

The ability to read a statement carefully in order to grasp the entire meaning is probably the most essential general process. It is not peculiar to mathematics, but probably no other subject furnishes such definite examples of the disastrous results of failure to comprehend the exact meaning of a statement. In the list presented in this chapter careful reading has not been included for the reason that it is uniformly necessary.

The number in parentheses following each of the general processes listed indicates the number of times this particular process or procedure is used in the exposition of the text and in the solution of the problems. In order to make clear the nature of the procedure, one illustrative problem has been quoted in each case with its page and number indicated in parentheses immediately after the problem.

1. The selection of the appropriate formula and the numerical substitution in it. (428)

> "The voltage across the terminals of a direct-current motor is 205 volts, and the current is 20 amperes. If the internal resistance is 0.5 ohm, what is the counter E.M.F. of the motor?" (462–20)

Formula: $V = E + Ir.$

Substituting: $205 = E + 20(0.5)$

$E = 195.$

This is a simple illustration of the most frequently occurring procedure (428 times, as indicated above) throughout the entire subject, unless we include the item of careful reading of the problem. In this case, the careful reading is really the essential thing in order to select the appropriate formula. The inherent difficulties in this type of procedure are in the reading fully as much as in the mathematics. The mere mechanical substitution of numbers

for letters should not be difficult, and yet there are many opportunities for errors in this part of the work. The resulting arithmetic computations indicated in later illustrations will be found to be more complicated than the type of practice work now generally advocated for the arithmetic of the grades. An examination of the majority of algebra textbooks will reveal a shortage of the type of numerical substitution that is required in physics formulas. Young [4] expresses his views as follows:

> Teachers of physics often find that pupils see no numerical meaning in the formulas of algebra, that they are not able readily and properly to interpret and apply the formulas in numerical instances. The numerical evaluation of algebraic expressions by the substitution of specific numbers for the letters cannot readily be overdone in beginning algebra. . . . The use of specific numerical values for the letters is useful not only throughout the secondary school but far beyond it. Besides its utility in clearing up the pupil's conception of the process itself, it is also valuable as a check against mechanical errors and inadvertencies.

2. Reduction of data to the proper units of measure before substitution in the appropriate formula. (154)

> "A gun shoots a shell weighing a ton with a muzzle velocity of 2250 ft./sec. Compute the energy in foot-pounds." (107–10)
>
> Formula: $W = 1/2\ mv^2$.
>
> Substituting: $W = 1/2\ \frac{2000}{32}\ (2250)^2 = 1.58 \times 10^8$.

The use of the C.G.S. system in the sciences necessitates a type of watchfulness on the part of the student that is very seldom required in high-school mathematics. If the weight and velocity in this problem had been stated in grams and in centimeters per second, these would have been substituted directly in the formula in the places of m and v and the answer would have been in the proper unit of measure (ergs). The science student is obliged to check the numerical values of all letters before he substitutes, in order to assure himself that they have been reduced, not to the same denomination as the other letters, but to the proper unit to fit the specific formula under consideration.

The most frequent source of error in classes in elementary statistics is this same question of the proper unit of measure. When computations are made from a frequency distribution by *the guessed average method,* the *correction* may be expressed in either step intervals or score units. In the formula for computing the

[4] Young, J. W. A., *The Teaching of Mathematics,* pp. 297, 298. Longmans, Green & Company, 1927.

mean, it is expressed in score units, whereas, in the formula for the standard deviation which is ordinarily used, it must be expressed in step intervals. The standard deviation may also be expressed in either step intervals or score units. In the formula for the correlation coefficient it is expressed in step intervals, but in the formula for the regression equation it is expressed in score units. It is very seldom that a pupil in algebra or geometry has any experience with problems that prepare him for this type of difficulty. The old rule for finding the number of board feet in a piece of lumber (multiply the length in feet by the width in inches by the thickness in inches, and divide by 12) is somewhat similar. Problems in trigonometry concerning degrees and radians provide a valuable experience.

3. Numerical substitution in a verbal law or definition. (104)

"The weight per cubic foot of a certain grade of pressed brick is 150 lb. Find the average specific gravity of the brick." (35–11)

The specific gravity of a body is defined as *the ratio of the weight of that body to the weight of an equal volume of water.* Therefore, the specific gravity of the brick is found by dividing 150 by 62.4. This type of substitution requires a careful reading of the problem and a careful reading of the law or definition. There are no difficulties in the mathematics involved. The difficulty arises when the law or relationship is implied instead of being stated definitely. Ligda [5] points out the fact that much of the difficulty in solving the verbal problems of algebra is due to the fact that pupils fail to recognize the underlying law, either stated or implied, which connects the given data. This law or relationship constitutes his *characteristic formula* which must always be discovered and made the basis of the solution.

4. The resolution of a problem into two or more component parts, each part being dependent upon the one preceding. (55)

"A ball is thrown horizontally with a speed of 50 ft./sec. from the top of a tower 150 ft. high. How far from the base of the tower will it strike the ground?" (65–17)

Formula:	$s = 1/2\ at^2.$
Substituting:	$150 = 1/2\ (32)t^2$
	$t = 3.06.$
Formula:	$s = rt.$
Substituting:	$s = 50(3.06) = 153.$

[5] Ligda, Paul, *The Teaching of Elementary Algebra,* pp. 117ff. Houghton Mifflin Company, 1925.

Naturally, the first effort of the student is to find a formula in which to substitute the given data in order to find the required distance. There is no hint that the first problem is to find t, and that this computed item is to be used to determine the final answer. The ordinary exercise or proposition in geometry involves just this sort of procedure where the student is required to go from a stated position to a required conclusion without being told the intermediate steps which he must take. However, the computation problems in which this occurs are not frequent. One such type in geometry is the following: "The circumference of a circle is 100 inches, find the area." Teachers find that pupils do not readily solve this sort of problem when they first meet it. This is probably due to the fact that they do not think of the intermediate step of solving for the radius or diameter. Problems in arithmetic and geometric progressions might be so stated that the pupil would be obliged to make an intermediate computation not called for before actually computing the required letter. Thus, "Find the sum of 11 terms of the progression 2,11,20. . . ." The usual formula $s = n/2(a + l)$ cannot be used until l has been found. The ordinary text, however, says "Find l and s." This deprives the pupil of the experience of discovering for himself when it is necessary to compute an intermediate item in order to make the required computation. More frequent situations in high-school mathematics where the pupil is obliged to make a computation not called for, before proceeding to *get the answer* would help him to adjust himself to this situation in physics.[6]

5. The resolution of a problem into two or more independent parts which are finally combined for the required answer. (6)

> "A photographic camera has a focal length of 6 in. How much farther from the lens is the image of an object 10 ft. away than that of an object 100 ft. away?" (664–1)
>
> Formula: $1/p + 1/q = 1/f.$
> Substituting: $1/120 + 1/q = 1/6$
> $q = 6.32$ (first)
> Also $1/1200 + 1/q = 1/6,$ $q = 6.03$ (second)
> Difference $= 0.29.$

This type of problem is not so frequent; nor is it so difficult. The wording of the problem usually indicates that two items are

[6] See number 10 for an illustration of a slightly different situation.

to be found and then combined in some manner. Of the six problems of this type found in Stewart's textbook, the two results were to be multiplied in one case, added in one case, and subtracted in four cases.

6. Transferring from the type or illustrative problem to the problem as stated, and then planning an analogous solution. (64)

> "If a man eats 1000 kg.-cal. in 20 min., at what horse power is he receiving energy?" (249–5)

The illustrative problem reads as follows:

> "A moderately active man should receive daily an average of about 3000 kg.-cal. of energy. He gets his energy from his food. To how many foot-pounds does this correspond?
>
> 3000×3090 ft.-lb. $= 9.27 \times 10^6$ ft.-lb."

Accordingly, in the above problem,

$1000 \times 3090 =$ number of ft.-lb.
1 horse-power $= 550$ ft.-lb./sec.

Therefore by *unitary analysis,* combining the illustrative problem and the definition of horse-power, the analogous solution is indicated as follows:

$$\frac{1000 \times 3090}{20 \times 60 \times 550} = 4.7 \text{ H.P.}$$

There may be a question concerning the advisability of following a *pattern* instead of establishing the solution directly from the specific laws and definitions previously stated. The above illustration is a very simple one. The writer found that in a great many cases he was wasting time in endeavoring to *think through* the solution from the fundamental laws and definitions instead of referring to a model problem analogous to the one stated. In Stewart's text it is very evident that a conscious effort has been made to put at the end of the chapter many problems which apply principles that have been illustrated by model problems. High-school pupils have ample experience in following models in geometry, and probably are more dependent upon *types* and *patterns* in algebra than would be ideal for developing independent thinkers, and yet a certain amount of this experience is essential. The subject of statistics again furnishes an excellent illustration. It is probably safe to say that ninety-five per cent of the students in educational statistics follow *models,* or *types,* or *patterns* to a large extent in *writing* their formulas and in making their com-

putations. The texts are written and the classes are taught by specialists in education. In fact, the students themselves are specialists in education. The student who is most expert in adapting himself to this method is apparently the most successful. It is the approved method with these experienced adults. In the light of these facts, the question arises whether or not the high-school pupil should be specifically and efficiently trained to follow step by step model or type problems.

7. A certain amount of generalization from the type or illustrative problem in order to transfer to the new problem, and then planning an analogous solution. (53)

> "500 gm. of aluminum at 90° C. are mixed with 200 gm. of water at 10° C. What is the resulting temperature if the specific heat of aluminum is 0.21 cal. per gm. per degree?" (234–5)

The illustrative problem reads as follows:

> "200 gm. of water at 10° C. are mixed with 500 gm. of water at 30° C. Compute the temperature of the mixture. Equating these,
>
> $$200(t-10) = 500(30-t)$$
> or $$t = 24.3° \text{ C.}"$$

The illustrative problem deals with two quantities of water, whereas this problem deals with aluminum and water. The generalization consists in appreciation of the fact that the loss of temperature in one equals the gain of temperature in the other. This coupled with an understanding of the meaning of specific heat enables us to write the new equation:

$$500(0.21)\ (90-t) = 200\ (t-10)$$
$$t = 37.5.$$

It is possible to solve problems mechanically by the method discussed under number 6 without actually comprehending the principles involved, but in order to generalize, and then establish an analogous solution, the student must understand the underlying principles.

8. Unitary analysis. (64)

> "A 25-watt lamp, vacuum type, is rated at 1.1 watts per candle power. What intensity of illumination in foot-candles is produced at a distance of 3 ft. from the lamp?" (605–1)

Formula: $$\text{Intensity of illumination} = \frac{\text{candle power}}{(\text{distance})^2}$$

Candle power must first be obtained: $25/1.1 = 22.727$.

$$\text{Therefore: Intensity of illumination} = \frac{22.727}{3^2} = 2.525.$$

So far as the solution is concerned, the following problems are identical:

(1) If it requires 1.1 watts for one candle power, how many candle power will 25 watts produce?
(2) If an automobile goes 1.1 miles in one minute, how long will it require to go 25 miles?
(3) If a marble costs 1.1 cents, how many marbles can Frank buy for 25 cents?
(4) If the amount of one dollar for two years is 1.1 dollars, how many dollars must be placed at interest for two years to amount to 25 dollars?
(5) If 1.1 gm. of a certain salt are required to produce 1 c.c. of a saturated solution, how many c.c. of saturated solution can be produced with 25 gm. of the salt?

Even though a student be able to answer such a question as number (3) without any difficulty, we are not justified in assuming that he can solve number (1) as it is stated in physics, number (2) as a *rate* problem, number (4) as a problem in *present worth* or *investments,* or number (5) as a problem in *solutions* in chemistry. With the proper analysis and organization of material, possibly all such problems could be presented as illustrations of the same principle. At present, these five problems are taught in five different years by five different teachers, each teacher having more or less difficulty in making clear the same underlying principle. One of the many opportunities of algebra as generalized arithmetic is to take such problems as these and to show that all of them fall under one fundamental principle.

9. Changing from one measure to another by means of a known ratio. (239)

> "The pressure in city gas mains is 2.5 in. of water. Find the pressure in pounds per square inch." (35–5)

From a table, the equivalent of one inch of water is found to be 0.0361 pounds per square inch. Therefore, multiply 0.0361 by 2.5 obtaining 0.090 pounds per square inch. All reductions from one unit of measure to another come under this heading. The difficulty is not in the mathematics, but in recognizing problems

in which it is necessary, and in knowing the ratio, or knowing where to find it.

10. Discarding irrelevant data. (14)

> "A 5-ounce baseball was dropped 557 feet from a window of the Washington monument and was successfully caught. Neglecting air effects, (*a*) What was its final velocity? (*b*) How long did it take to fall?" (65–14)
>
> (*a*) Formula: $v = 32t$.
> (*b*) Formula: $s = 1/2\, at^2$.
> Substituting: $557 = 1/2\,(32)\,t^2$
> $t = 5.90$.
> Substituting in (*a*): $v = 32(5.90) = 188.8$ ft./sec.

The first inclination is to endeavor to use the given weight of 5 ounces. This particular problem is still more confusing because we know that it actually does make a difference whether the ball weighs 5 ounces or is a hollow rubber ball weighing one ounce. The phrase "neglecting air effects" furnishes another illustration of the importance of reading the problem carefully. This makes the 5 ounces an irrelevant fact that must be discarded.

When in the great majority of the problems, all available data are utilized, the most natural reaction of the student is to attempt to use every numerical item. It is true that in geometry the student has ample experience in selecting the significant facts and in discarding all other facts. In problems of a computational nature, however, very few have ever appeared in algebra or geometry where numerical data have been given when they were not essential for the solution of the problem. Daily [7] concluded from his study that:

> Textbooks in algebra give few problems which contain more or less data than are needed to solve them, and teachers generally do not use such problems. The introduction of problems containing superfluous or insufficient data is disconcerting to pupils who have been accustomed to problems containing only the necessary data. . . . A small amount of drill is needed to teach pupils to choose the necessary data from among superfluous data. . . .

Since we know that such problems do occur in physics, it is evident from Daily's conclusions that pupils should be and can be better prepared for them.

[7] Daily, B. W., *Ability to Select Data in Solving Problems*, p. 81. Contributions to Education, No. 190. Bureau of Publications, Teachers College, Columbia University, 1925.

This same problem illustrates another interesting situation. In number 4, discussed on page 60, we had an illustration of the necessity of taking an intermediate step, of which there was no hint in the question, in order to obtain an item essential for the computation of the final answer. In *this* problem, it is necessary to make the computation called for in the second part in order to find the item needed for substitution in the formula used in the first part. Such problems could be devised in algebra.

11. Making use of a fact, technique, or procedure that has been suggested by a preceding problem. (27)

> "A large steam plant used 1.5 lb. of coal for each horse-power hour, burning coal which had a heat of combustion of 14,500 B.T.U./lb. What percentage of the total energy was obtained?" (324–3)

In the preceding problem, one horse-power hour has been found to be $550 \times 3600 = 1{,}980{,}000$ foot pounds. The preceding problem has also reduced foot-pounds to B.T.U. by multiplying by 1.286×10^{-3}.

$$198 \times 10^4 \times 1.286 \times 10^{-3} = 2546.28 \text{ B.T.U.}$$
$$14{,}500 \times 1.5 = 21{,}750 \text{ B.T.U. from coal.}$$

Heat of combustion was also used in the preceding problem. The only new feature of the problem is the question of percentage.

$$2546.28/21{,}750 = 11.7\%$$

The device of presenting problems which involve facts or methods developed in preceding problems is not peculiar to this textbook. Practically all geometry texts are developed on this plan. Much of the algebra is presented in the same manner. The plan is more pronounced in geometry and in the development of new material in algebra than it is in the exercises and drill problems.

12. Practical interpretation of a law or relationship stated symbolically. (24)

> "When a rod of length l and radius r is twisted by a torque L applied at the free end of the rod, the angle through which the end is twisted is
>
> $$\phi = \frac{2lL}{n\pi r^4}$$
>
> where n is the modulus of rigidity of the material of which the rod is made." (200)

From this equation, the following laws are evident:

(1) The angle of twist is proportional to the torque.
(2) The angle of twist is proportional to the length of the rod.
(3) The angle of twist is inversely proportional to the fourth power of the radius.
(4) The angle of twist is inversely proportional to the modulus of rigidity.

The importance of the ability to interpret algebraic formulas and to recognize relationships has been so frequently emphasized that it seems appropriate to quote three authorities. Young [8] gives various type formulas and questions for drill, in connection with the following statement:

It is equally important, conversely, to state in words the content of equations and formulas, and this likewise needs continual and gradual drill. . . . This constant passage back and forth from the compact form of the equation (an algebraic shorthand) to the expanded verbal form will consume an appreciable fraction of the time, but much of it will be regained by the rapid solution of the problems themselves.

The Report of the National Committee [9] also gives several illustrative formulas and questions with the statement:

The study of formulas as such, aside from their numerical evaluation, is becoming of more and more importance. The actual uses of algebra are not to be found solely, nor even principally in the solution of numerical problems for numerical answers. In such formulas as those for falling bodies, levers, etc., the manner in which changes in one quantity cause (or correspond to) changes in another, are of prime importance, and their discussion need cause no difficulty whatever. The formulas under discussion here include those formulas of pure science and of practical affairs which are being introduced more and more into our texts on algebra. Whenever such a formula is encountered, the teacher should be sure that the students have some comprehension of the effects of changes in one of the quantities upon the other quantity or quantities in the formula.

Thorndike [10] makes the following comment:

The understanding of equations, as the expression of relations, goes straight to the heart of all applied mathematics, showing the formula and equation as the story of a rule or law which certain events in nature follow or approximate; it introduces the most important idea of mathematics, that of quantitative dependence or functionality; it is a vital and potent review of the principle that algebra tells what will happen to *any* number under certain conditions; it furnishes a principle of organization for graphics; it

[8] Young, *op. cit.*, pp. 310, 311.
[9] National Committee Report, pp. 66, 67.
[10] Thorndike, E. L. and others, *The Psychology of Algebra*, p. 107. The Macmillan Company, 1923.

furnishes the treble parallelism between certain important relations, certain graphs and certain equations which will arouse respect for algebra.

As a result of all that has been said and written in favor of more practice in interpretation of formulas, and of the importance of appreciation of the functionality concept as expressed by formulas, many of the recent textbooks have shown a marked improvement in this respect. Unfortunately, the tendency of the *standardized tests* is to emphasize the importance of the manipulative skills rather than the more desirable outcomes from training in mathematics. The reason is evident. It is a comparatively easy task to devise a test of the ability to manipulate algebraic symbols. It is not so easy to determine the ability to analyze, generalize, or interpret.

13. Interpretation of a negative root of an equation that has been established to solve a practical problem. (13)

> "Find the position of the image formed by a concave mirror of radius 30 cm. when the object is placed 10 cm. from the mirror." (622–1)
>
> Formula: $1/p + 1/q = 2/R.$
> Substituting: $1/10 + 1/q = 2/30$
> $q = -30.$

The image is therefore 30 cm. back of the mirror, and is a virtual image.

Abstract problems for drill in algebraic equations frequently have negative roots. The negative root is so rare in the so-called *verbal problem,* however, that the pupil usually considers it with a degree of suspicion. The work with mirrors and lenses emphasizes the importance of the pupil's appreciation of the oppositeness of signs in results. Pupils enjoy the interpretation of a negative result in a cistern problem where it turns out that the water is running through the pipe in the opposite direction to that originally assumed, especially when they are convinced by numerical substitution that the interpretation is perfectly consistent.

14. Translation of a verbal law or definition into a mathematical formula. (85)

> "It is found by experiment that the rate of flow of heat is (1) proportional to the difference of temperature $t_1 - t_2$ of the two faces (of a slab of some conducting material with faces AA and BB); (2) inversely proportional to the thickness d of the slab; (3) pro-

portional to the area of the faces AA and BB. These facts may be stated by the following equation:

Rate of flow of heat in calories per second $= k \frac{t_1 - t_2}{d} \times$ area."
(253)

The ordinary verbal problems of algebra provide for expressing relationships by means of algebraic equations of a certain type. Many of the texts have preliminary or introductory practice questions such as the following: "If p pencils cost c cents, what is the cost of one pencil? If a train goes m miles in k hours, what is its rate per hour?" Such questions are excellent in helping the student to appreciate the generality of algebra, and in preparing him for the type of problems that he finds in the text. The problems in science deal quite largely with the statements *varies as, varies inversely as, is directly proportional to,* and *is indirectly proportional to.* The majority of the eighty-five cases that have been tabulated here are of this type. There is probably a greater discrepancy between the number of times the student of physics has occasion to use this technique and the number of opportunities he has met for practicing it in algebra than in any other process.

The student's knowledge of proportion is derived chiefly from geometry. If he has had any experience whatever with variation it has been a very brief consideration of the subject at a time far removed from the geometry. The two concepts are not associated in the mind of the average high-school graduate. To him, proportion is that part of geometry that is taught just before he studies similar triangles. A proportion is defined as "the equality between two ratios." In other words, students are taught that in order to write a proportion they must have four quantities. When the subject of variation is presented, two quantities are used. Thus, the cost of building a walk of a given width varies as the length. That is, if the length is doubled, the cost is doubled. It is true that some pupils have been fortunate in having these two subjects presented as related concepts, but many have completed high school without realizing that proportion and variation are two phases of the same subject.

It is not surprising that the student who has learned to think of proportion and variation as two distinct subjects, is confused when he is plunged into physics and meets the following state-

ments concerning the volume of a given mass of gas kept at a constant temperature:

(1) "The pressure varies inversely as the volume,

$$pv = p'v' = \text{constant.}"$$

The mathematical statement of the law does not bear any resemblance to the student's conception of variation. He probably would write it $p \propto \frac{1}{v}$, or $p = k\frac{1}{v}$, or possibly $pv = k$.

(2) "The product of the pressure and the volume is constant,

$$p \cdot v = p_1 \cdot v_1 \quad \text{or} \quad \frac{p_1}{p} = \frac{v}{v_1}."$$

In this case, the verbal statement mentions *constant* but the mathematical statement does not. In the preceding statement, the mathematical equation contained the word *constant,* but the verbal statement did not.

(3) "The volume is inversely proportional to the pressure,

$$p_1v_1 = p_2v_2 = \text{constant.}"$$

Here again, the mathematical statement has the word *constant* while the verbal statement does not. Furthermore, the verbal statement leads the pupil to expect a proportion, but none appears in the mathematical statement of the law. The textbook which presents Boyle's Law in the form indicated in (2), makes the following statement concerning force and acceleration: "It is found by experiment that the acceleration produced in any mass is strictly proportional to the force which causes that acceleration; that is, $F \propto a$." Here the verbal statement suggests *proportional,* and the mathematical statement uses the sign for *varies as.*

These examples have been quoted from three of the texts listed in Chapter III, and show how the student of physics must be prepared to use these various verbal and algebraic forms as equivalent. Ohm's Law, Boyle's Law, the laws concerning friction, illumination, and acceleration, in fact, all laws concerning relationships that are directly or inversely proportional make the verbal statement relative to two quantities. The mathematical statement of the law is sometimes expressed with four letters, and sometimes with two.

If students are to be prepared to meet this situation in physics,

such statements as the following will be helpful: circumferences of circles vary as the diameters; circumferences of circles are proportional to the diameters; the circumference of a circle varies as the diameter; $\frac{C}{D} = \frac{C'}{D'} = \pi$; $C \propto D$; $C = \pi D$; the area of a circle is proportional to its radius squared; the area of a circle varies as the square of its radius; in rectangles having a constant area, the bases are inversely proportional to the altitudes; in rectangles having a constant area, the bases vary inversely as the altitudes; if a rectangle has a fixed area, its altitude is inversely proportional to its base; $\frac{b}{b'} = \frac{a'}{a}$; $ba = b'a'$; $b \propto \frac{1}{a}$; $ab = K$; $b = K\frac{1}{a}$.

Doubtless many teachers think they are teaching pupils to recognize all of these forms as equivalent. Perhaps they are. Other teachers will insist that the true proportion must deal with four quantities, and that therefore the pupils should be taught to say "In any two circles, the two diameters are proportional to the two circumferences" or "the ratio of the circumference to the diameter of one circle equals the ratio of the circumference to the diameter in any other circle"; that furthermore, the statement "the circumference of a circle is proportional to its diameter" is careless, inaccurate, and not to be tolerated in high-school mathematics. Perhaps they are right. The fact remains that this form of statement predominates in physics, and it is assumed in the physics class that the student can transfer from any one form to any other form without difficulty.

15. Interpretation of a formula as certain values or limitations are assigned to a letter or letters of the formula. (14)

> "If the absolute temperature of a black body is doubled by what factor will the amount of radiated energy be increased?" (265–9)
>
> Formula: $E = cT^4$.
> Doubling T: $E = c(2T)^4 = c\,16\,T^4$.
> Therefore, E becomes 16 times as great.

Problems 11 and 13 in the test discussed in Chapter VI further illustrate this type of reasoning.

The student in statistics who has had no experience in interpreting formulas is seriously handicapped. It may be asserted with more or less justification that the mathematical development of many of the statistical formulas is not essential for the student

in education. Regardless of divergent opinions on this question, it is evident from the illustrations that follow that he should be able to recognize the relationships of the letters in the formula. In the formula $P.E._r = \frac{1 - r^2}{\sqrt{N}}$, the student must see readily that an increase in N decreases the fraction, and that an increase in the absolute value of r (which is between $+1$ and -1) decreases the fraction.

In the formula $P.E._M = .6745\frac{S.D.}{\sqrt{N}}$, it should be evident that the $P.E.$ increases as the $S.D.$ increases, and that the $P.E.$ decreases as N increases.

If $r = \sqrt{b_1 b}$ (where r is between $+1$ and -1), the student should be able to answer the following questions concerning b_1 and b:

(1) Can both be smaller than 1?
(2) Can both be larger than 1?
(3) Can they be opposite in sign?
(4) Can both be negative?
(5) Can either be larger than 1?

Such problems as those given in Thorndike's [11] I.E.R. Algebra Tests emphasize this type of training.

16. Interpretation of the meaning of the negative value of a letter substituted in a formula. (5)

> "If a represents the acceleration, t the time, u the initial velocity, and v the final velocity, the gain in velocity is at, and the final velocity is $v = u + at$. . . . In case the body is losing velocity at a constant rate a, in t seconds, the loss in velocity will be equal to at. Hence the resultant velocity at the end of t seconds is $v = u - at$." (56 and 57)

The tabulation of the number of times the idea of oppositeness of positive and negative is used appears in Chapter IV. This illustration is not merely the idea of oppositeness, but it also involves the ability to express an opposite force or quantity by using the same letter in a formula with its sign changed.

17. Generalization from a numerical problem to a literal formula. (3)

[11] Thorndike, *op. cit.*, p. 183.

"In the problem of a body thrown vertically downward with an initial velocity of 40 feet per second, how should one proceed to calculate the distance travelled in 3 seconds? This can be solved in a rather simple way: the initial velocity is 40 feet per second, and the final velocity is $40 + (32 \times 3) = 136$ feet per second. The average velocity will be half the sum of the initial and final velocities, or

$$\text{Average velocity} = 1/2\ (40 + 136) = 88 \text{ ft./sec.}$$

But the distance travelled is equal to the average velocity times the time, or

$$\text{Distance} = 88 \times 3 = 264 \text{ ft.},$$

which is the desired result. By following this method a general law for computing the distance can be derived. Let the initial velocity be represented by u, the final velocity by v, and the average velocity by $\bar{v}$. The average velocity, as before, will be half the sum of the initial and final velocities, or

$$\bar{v} = 1/2\ (u + v).$$

Substituting in this the value of v $(v = u + at)$,

$$\bar{v} = 1/2\ (u + u + at) = u + 1/2\, at.$$

The distance is given by the equation $s = \bar{v}t$,

hence
$$s = (u + 1/2\, at)\, t$$
$$= ut + 1/2\, at^2,$$

which is the general law for the distance s traveled in a time t." (56, 57)

This entire development is reproduced from the text because it is an excellent illustration of the process of generalization. Ability to generalize in this manner is extremely important. It is an ability that is not easily acquired. The teacher of algebra who is successful in teaching verbal problems is constantly employing this method. The solution of the verbal problem is of minor importance as compared with the development of the ability to change from the simple numerical illustration to the generalized algebraic statement.

18. Mathematical statement of a law which has not been expressed in words. (63)

"The law for computing the total resistance, R, of three resistances r_1, r_2, and r_3 connected in parallel is

$$1/R = 1/r_1 + 1/r_2 + 1/r_3." \quad (448)$$

This is a very common practice in physics. No attempt is made to state the law in words since the algebraic equation is much more

concise. The importance of mathematics as a language by which laws and relationships can be concisely expressed cannot be overemphasized in high school.

19. Using a verbal rule as a stimulus to a process carried on in imagination. (7)

> "A north-seeking pole is placed just below an east and west wire carrying a current eastward. What is (*a*) the direction of the current-magnetic-field-force on the wire? (*b*) the force on the pole? (Neglect the effect of the earth's magnetic field.) (493–4)

The *right-hand rule* is "When a person, actually or in imagination, grasps a wire with his right hand, with the thumb lying along the wire and pointing in the direction of the current, the fingers will point around the wire in the same direction as the magnetic field." Therefore, in this case, the force on the pole is northward, and since the force on the wire is the opposite, it is southward.

20. Use of the index of the textbook to locate laws and definitions previously stated. (27)

In the problem just quoted in number 19, it was necessary to use the so-called *right-hand rule*. It had not been used for some time, and had therefore been forgotten. A glance at the index indicated at once that it appears on page 411. Many pupils do not appreciate the value of an index unless they are actually trained in its use. Geometry offers an excellent opportunity to insist upon the use of the index every time a pupil forgets a definition which he needs in a proof.

21. Solution of a problem for the ratio between two quantities when the value of neither quantity is known. (19)

> "What is the fractional change in the volume of a glass ball produced by a pressure of 3000 lb./in.^{2}?
>
> ($K = 6 \times 10^6$ lb./in.^{2})." (202–10)

Formula: $$K = \frac{pv}{\text{change in volume}}.$$

Substituting: $$6 \times 10^6 = \frac{3000\,v}{\text{change in volume}}.$$

Whence: $$\frac{\text{change in volume}}{v} = \frac{3000}{6 \times 10^6} = 0.0005.$$

Solving for a ratio between two letters occurs frequently in the development of formulas, but appears only once in the problems. The importance of drill in this work was discussed in Chapter IV.

22. Checking problems for reasonableness of results. Theoretically, pupils should check every problem roughly to see whether the answer seems reasonable. This verification is emphasized in most high-school classes. Physics problems should be checked in this manner so far as possible. We must admit, however, that many of these problems deal with quantities so far removed from the actual experiences of the students that it is impossible to judge as to the reasonableness of the answers. The following is a good illustration:

> "Compute the natural frequency of oscillation of a circuit consisting of a 0.015-microfarad condenser in series with an inductance of 0.0002 henry." (591–1)

Formula: $f = \frac{1}{2\pi\sqrt{LC}}$.

Substituting: $f = \frac{1}{6.2832\sqrt{(0.0002)\ (0.000000015)}}$

$= 91{,}889 \text{ sec.}^{-1}$

The physics student is not likely to know whether or not this answer is reasonable.

There are, however, many problems in physics in which a consideration of the conditions of the problem will give a very definite idea of the size of the answer. Numbers 4, 5, 7, and 10 of this chapter illustrate cases in which there would be no possibility of misplacing the decimal point if the student would consider the reasonableness of the answer. Frequently, an inspection of the equation is sufficient to estimate the size of the answer even though the student may not be familiar with the conditions of the problem that gave rise to the equation. Such an inspection of numbers 1, 2, 5, and 8 in the test discussed in Chapter VI would have prevented many of the errors that occurred in these problems.

On account of the predominance of integral results in the problems of algebra, many pupils are satisfied with such results, and are inclined to doubt the reasonableness of fractional results. Symonds [12] calls attention to this unfortunate situation in the following statement: "It is a bad habit to fix—that of judging the correctness of an answer by its *coming out even*—but no change can be made using our present textbooks unless teachers

[12] Symonds, P. M., "The Psychology of Errors in Algebra," *The Mathematics Teacher*, Vol. XV, No. 2, p. 96. (February, 1922.)

demand it." Naturally, this criterion would not be applicable in physics. Very few of the problems are fixed in physics as so many seem to be in algebra. Of the 572 problems in the textbook studied, 30 were non-numerical, leaving 542 that had numerical results. Of these 542 results, 210 were integral and 332 were fractional.

CHAPTER VI

TEST ON FORMULAS

The statement is frequently made in Chapter IV that certain algebraic manipulations are *unusual* or *different*. The implication that high-school pupils will find these manipulations difficult, because they are *unusual* may be questioned. In fact, the opposite opinion was almost unanimously expressed by two groups of experienced teachers of secondary mathematics before whom these examples were discussed.

In order to obtain objective evidence concerning the relative difficulty of the various formulas listed in Chapter IV, the writer devised a simple test. All tabulations of symbolic manipulations and interpretations occurring in the study of Stewart's text, including the solution of all of the problems, were scrutinized carefully in order to obtain the representative types. Twenty-six examples were thus selected. This test was given to a class of forty-eight teachers and prospective teachers of high-school algebra, and they were allowed thirty-eight minutes in which to complete it. About two-thirds of the class were able to complete it in this time. These twenty-six examples with the answers obtained by the members of the class were then discussed by a seminar group of ten experienced teachers of mathematics for the purpose of reducing the number so that an ordinary high-school class could answer the questions in a forty-five minute period. The eighteen examples in the test which appears on pages 78 and 79 were then selected.

The following explanatory note was sent to teachers in fourteen schools who had agreed to give the test:

The test which you so kindly consented to give is being sent to you under separate cover with a self-addressed stamped envelope for the return of the papers.

This test does not purport to be a survey of the material that should be covered in high-school algebra, but merely involves the types of problems that actually occur in college freshman physics.

The test is planned for pupils who have had at least three semesters of

algebra. It has been estimated that it should require forty-five minutes, but inasmuch as it is not a speed test, there are no objections to allowing a longer time. If you will make a note on the papers indicating the time allowed, it will be greatly appreciated.

CONGDON ALGEBRA TEST

Read each exercise carefully; then do as you are told, remembering that accuracy is more important than speed. If you are not sure what to do in any case, do not waste time, but go on to the next exercise. Your score will be determined by the number of correct answers. Do your figuring on the margin of this paper.

1. Given $\frac{1}{p}+\frac{1}{q}=\frac{2}{R}$. If $p=60$ and $R=30$, then $q=$ .....
2. Given $I=\frac{e}{R+\frac{r}{N}}$. If $e=1.5$, $R=4$, $r=3$, and $N=2$, then $I=$ .....
3. Given $P=Fv$. If $P=5500$ and $v=\frac{88}{3}$, then $F=$ .....
4. Given $T=\frac{44}{7}\sqrt{\frac{l}{g}}$. If $T=1.1$ and $g=980$, then $l=$ .....
5. Given $V=E+Ir$. If $V=20$, $E=2.2$, $I=5$, and $r=0.1+R$, then $R=$ .....
6. Given $\frac{I}{i}=\frac{R}{r}$. If $R=0.1$ and $I+i=100i$, then $r=$ .....
7. Given $\frac{1}{p}+\frac{1}{q}=\frac{2}{R}$. If $R=10$ and $\frac{p}{q}=6$, then $p=$ .....
8. Given $0.8=\frac{25}{25-k}$. Then $k=$ .....
9. Write the value of $\frac{1.4\times10^{23}}{7.0\times10^{14}}$. $=$ .....
10. Using the initial letters of the words, write the mathematical formula which says "illumination varies inversely as the square of the distance." .....
11. If p is always less than $\frac{1}{2}R$, what must be the sign of q in the equation $\frac{1}{q}=\frac{2}{R}-\frac{1}{p}$? .....
12. If $E=cT^4$, tell exactly what effect will be produced upon E by doubling T. .....
13. If $\frac{W}{F}=\frac{2R}{R-r}$, and if r is less than R, describe the change in the value of $\frac{W}{F}$ as r and R get very near the same size. .....

14. Given $IE = I^2R + Ie$. Then $I = \ldots\ldots$

15. Given $Q = N_2 \frac{\phi}{R_2}$. Then $\phi = \ldots\ldots$

16. Given $\frac{R-p}{q-R} = \frac{p}{q}$. Then $\frac{1}{p} = \ldots\ldots$

17. Given $\frac{m'g}{L} x = \frac{4\pi^2 m' x}{T^2}$. Then $T = \ldots\ldots$

18. Given $F = \frac{W}{2R}(R - r)$. Then $\frac{W}{F} = \ldots\ldots$

More than one thousand tests were sent to schools in New York, New Jersey, Michigan, Ohio, and Nebraska. Most of the schools, however, were in the vicinity of New York City. Eight hundred fifty-nine papers were returned. Inasmuch as the instructions say nothing concerning the simplification of results, fractional answers in any form were accepted provided there were no errors. Number 17, for example, was scored as *right* even though the $4\pi^2$ or the g was left under the radical sign. It was also scored *right* if m' and x were left in the answer. Number 10 was considered *right* if it was expressed as a proportion, or an

TABLE XVI

PER CENT OF CORRECT ANSWERS FOR EACH PROBLEM IN EACH SCHOOL

PROBLEM	SCHOOL													
	A N=56	B N=39	C N=41	D N=19	E N=14	F N=180	G N=75	H N=111	I N=42	J N=11	K N=11	L N=110	M N=104	N N=46
1	89	74	44	89	86	84	76	77	60	100	45	75	84	78
2	89	72	63	89	86	91	88	79	60	100	45	73	78	87
3	64	51	27	68	57	72	73	71	52	18	55	55	61	57
4	5	8	0	21	7	16	20	14	14	0	18	13	7	11
5	70	49	22	58	57	67	68	57	50	27	27	58	55	59
6	23	3	5	53	36	28	32	22	14	18	9	18	28	24
7	25	3	0	42	14	28	39	14	21	27	0	11	40	24
8	61	28	27	42	29	50	52	41	26	45	18	36	61	57
9	34	36	10	58	64	32	32	35	43	45	36	32	21	43
10	9	49	0	42	14	5	7	31	10	9	9	20	7	4
11	30	13	12	32	21	34	27	23	10	55	27	22	33	50
12	7	0	2	16	14	7	12	4	5	18	9	3	8	11
	43	77	41	26	14	41	51	19	17	27	9	14	47	59
13	43	46	24	37	14	56	51	32	29	45	18	25	68	59
14	27	8	2	37	36	28	23	23	29	27	0	20	26	33
15	36	15	27	42	7	38	33	41	62	27	9	36	29	67
16	23	5	2	16	14	11	25	14	2	0	9	0	14	26
17	11	18	10	47	21	28	35	33	41	27	36	22	37	48
18	27	15	12	37	29	29	27	24	48	36	27	15	27	37
All	37	27	16	46	34	39	40	35	32	35	22	30	38	43

NOTE: The table should be read thus: School A sent 56 papers; of the answers to problem number 1, 89 per cent were correct; 89 per cent of the answers to problem number 2 were correct; . . . and finally, 37 per cent of all the problems from School A were correct.

equation with k, or with the sign for *varies as*. In number 12, the desired response is "16 times as great." This is the response tabulated as correct unless otherwise explained in connection with the tabulation.

Table XVI shows the per cent of correct answers for each of the eighteen questions in each of the fourteen schools A, B, . . . N. The number under each letter indicates the number of papers scored from that school. The last number in each column represents the per cent of *correct* answers in the school. This per cent was computed from the original data, and not from the partial per cents in the column. The second row under number 12 is the per cent correct including the response "increases it" as correct.

TABLE XVII

SUMMARY OF NUMBER AND PER CENT OF PROBLEMS CORRECT, OMITTED, AND WRONG

PROBLEM	NUMBER WRONG	NUMBER OMITTED	NUMBER CORRECT	PER CENT WRONG	PER CENT OMITTED	PER CENT CORRECT	PER CENT CORRECT
1.....	182	11	666	21.2	1.3	77.5	78.5
2.....	158	9	692	18.4	1.0	80.6	81.4
3.....	295	33	531	34.3	3.8	61.8	64.3
4.....	306	449	104	35.6	52.3	12.1	25.4
5.....	336	28	495	39.1	3.3	57.6	59.6
6.....	257	404	198	29.9	47.0	23.1	43.5
7.....	252	409	198	29.3	47.6	23.1	44.0
8.....	387	82	390	45.1	9.5	45.4	50.2
9.....	285	292	282	33.2	34.0	32.8	49.7
10.....	318	422	119	37.0	49.1	13.9	27.2
11.....	318	303	238	37.0	35.3	27.7	42.8
12.....	612	190	57	71.2	22.1	6.6	8.5
	356	190	313	41.4	22.1	36.4	46.8
13.....	157	324	378	18.3	37.7	44.0	70.6
14.....	360	295	204	41.9	34.3	23.7	36.2
15.....	298	245	316	34.7	28.5	36.8	51.5
16.....	314	441	104	36.6	51.3	12.1	24.9
17.....	200	409	250	23.3	47.6	29.1	55.6
18.....	195	439	225	22.7	51.1	26.2	53.6

NOTE: This table should be read as follows: Problem No. 1 was solved incorrectly by 182 pupils, omitted by 11, and solved correctly by 666; it was incorrect on 21.2 per cent of all the papers, omitted on 1.3 per cent of the papers, and correct on 77.5 per cent of the papers; and finally, of all (848) who tried the problem, 78.5 per cent obtained the correct result.

Table XVII is a summary for each problem showing the number and the per cent that were wrong, omitted, and correct. The last column shows the per cent of those tried which were correct.

Table XVIII shows the number of pupils who solved 18 problems correctly, 17 problems correctly, and so on. The mean of

TABLE XVIII

FREQUENCY OF NUMBER OF PROBLEMS CORRECT

Number of Problems Correct	Number of Pupils
18	0
17	1
16	3
15	1
14	8
13	21
12	19
11	39
10	49
9	61
8	86
7	102
6	105
5	102
4	97
3	75
2	54
1	23
0	13
Total	859

this distribution is 6.84 ± .07. The median is 6.62 ± .09, and the standard deviation is 3.14 ± .05.

The last row in Table XVI shows that the highest per cent of accuracy in any school is 46 per cent, and the lowest is 16 per cent. (These two schools were outside the state of New York.) The first row for number 12 is used in computing these per cents.

Table XVII shows the order of difficulty beginning with the easiest to be as follows: 2–1–3–5–8–13–15–9–17–11–18–14–(6, 7)–10–(4, 16)–12. Numbers within parentheses are equal in difficulty. If the last column is used, the order is slightly changed: 2–1–13–3–5–17–18–15–8–9–7–6–11–14–10–4–16–12.

The same test was given to a class of sixty teachers and pros-

pective teachers of algebra.[1] Following is the arrangement in the order of difficulty: (2, 3)–1–5–6–9–(13, 17)–(8, 14, 18)–15–7–11–4–12–16–10.

It should be noted that in all three arrangements, the same four problems are at the end of the list. They are in the same order for both lists for the high-school pupils. According to Table XVII it is evident that no school did well on numbers 4, 12, and 16. In the case of number 10 there is one exception. School B, in comparison with the other schools, did very well on number 10; and in comparison with other problems, number 10 was apparently easy for school B, there being but three problems with a better score.

At the other end of the distribution there is also general uniformity in numbers 1, 2, 3, and 5. In the distribution according to the per cent correct of those tried, number 13 is moved considerably nearer the end. This is to be expected, for there are not the chances for numerical errors in this. If the pupil thought he knew the answer, he put it down; otherwise, he omitted it. The scores on numbers 1 and 2 are relatively high for each of the fourteen schools. There is somewhat more variability in numbers 3 and 5.

A study of the record of the *Number Omitted* reveals some significant facts. The directions say "If you are not sure what to do in any case, do not waste time, but go on to the next exercise." Number 4 was omitted by 449 pupils—the greatest number that omitted any one problem. This indicates that pupils doubted their ability to solve this problem more than any other one in the list. Many of the omissions near the end of the list are due to lack of time, but omissions near the first of the list are clearly due to lack of confidence in ability. Numbers 6 and 7 were not omitted as often as were some near the end, but it is evident that they were also omitted because pupils did not know at once how to solve them. If 409 pupils omitted number 7 and only 82 omitted number 8, the only explanation is that they did not know how to solve number 7, and, furthermore, they knew they did not know how. Many of the omissions in the latter half of the list may be attributed to lack of time rather than to lack of ability.

The remainder of this chapter is devoted to a discussion of the

[1] This was a different group from the group of teachers mentioned in the first part of this chapter.

separate problems in the test in the hope of pointing out some of the type errors and difficulties that may be avoided by a changed emphasis in the teaching in high school.

Problem 1

This problem involves neither difficult algebra nor difficult arithmetic. The pupils tested were not afraid of it. Only 11 omitted it. The numerical substitution in the proper formula results in the following:

$$\frac{1}{60}+\frac{1}{q}=\frac{2}{30}$$

Of the 182 pupils who missed this problem, 41 obtained 1/20 for the answer. This error occurred more frequently than any other. There were 28 who reported the answer as -56. This is due to a failure to include q in the multiplier of the right-hand side of the equation when clearing of fractions, thus obtaining $q+60=4$, whence $q=-56$. An answer of 60 was obtained by 17 pupils, 30 by 11 pupils, and 20 by 5 pupils; the remaining 80 pupils had a variety of answers.

Problem 2

The substitution in this problem results in a complex fraction to be simplified. It might be classified as a problem in arithmetic rather than in algebra. Two type errors were frequent. Fourteen pupils obtained 3/22 as an answer. This was the result of dropping the denominator 2 when adding 4 and 3/2. There were 6 who inverted the answer, obtaining 11/3 instead of 3/11. Although these are arithmetic errors they are identical with the type of error frequently made in algebra. Proper drill in either the arithmetic or the algebra should improve the other. The other errors were scattered. It should be recalled that this is the easiest problem in the set, only 158 pupils missing it.

Problem 3

There were no outstanding types of errors in this problem. It deals with larger numbers, and therefore gives many more opportunities for arithmetical errors. Inverting the answer and misplacing the decimal point were among those most frequent.

Problem 4

This has been mentioned as the problem that was omitted by the greatest number of pupils. This is evidence that the pupils

had not had sufficient drill on this type of problem to have any confidence in their ability to solve it. Of those who had the courage to attempt it, only 25.4 per cent were successful. Misplacing the decimal point was the most common error. This indicates that the pupils did not take the trouble to check the answer roughly to see whether their results were reasonable.

Problem 5

The numerical substitution in this problem results in the following equation:

$$20 = 2.2 + 5(0.1 + R)$$

The most frequent error was in neglecting to multiply R by 5. The answer would then be 17.3. There were 32 who obtained this answer. Misplacing the decimal point was next in frequency. The correct answer inverted occurred several times.

Problem 6

This is another of the problems that was frequently omitted. Whereas only 28 omitted number 5, there were 404 who did not put down an answer for number 6. It does not resemble the ordinary problems of algebra. After substituting the numerical value of R, there are still three unknown letters with only two equations. It is not a type in which the pupil has been drilled. The arithmetic is not complicated. The solutions attempted seemed to be largely experimental. There were no outstanding types of errors.

Problem 7

This problem seems to be the equivalent of number 6 in difficulty. The fact that practically the same number omitted it indicates about the same degree of doubt on the part of the pupils concerning the method of solution. Exactly the same number missed it. It is sufficiently different from the ordinary *simultaneous equations* to cause pupils to attempt all sorts of methods. The only error that was frequent was putting down the value of q instead of that of p.

Problem 8

Apparently this problem appealed to the pupils as a familiar friend. Only 82 omitted it. It closely resembles the ordinary problems of algebra. However, it must have been deceiving. There is no apparent catch in the sign, but for some reason, 108

obtained 6.25 instead of -6.25 for the answer. The decimal point was misplaced by 29 pupils.

Problem 9

The first real evidence of guessing appears in this problem. Three pupils put down π as the answer. The confusion concerning the meaning and use of standard form numbers is shown by the frequent appearance of the two results 20^8 and 2^8 instead of $.2 \times 10^8$. Many of the errors in this problem are in misplacing the decimal point. Some of the time usually spent on complicated problems in the theory of exponents might profitably be used with standard form numbers.

Problem 10

It is evident from the results that many classes in which this test was given had not discussed variation. The number that omitted this problem (442) and the number that put down the initial letters *i, v, i, a, t, s, o, t, d* indicate quite clearly that they did not understand the question. On the other hand, the frequency of the answer $i = 1/d^2$ shows that many were sufficiently familiar with the concept to confuse the two forms $i = k/d^2$ and $i \propto 1/d^2$.

Problem 11

It would seem that a pure guess would be correct 50 per cent of the time in this problem. The sign of q is either $+$ or $-$. Of those who did not omit it, only 42.8 per cent put down the correct sign. These figures force us to the conclusion that there were not many of the 859 pupils who really knew much about this problem.

Problem 12

The wording of this problem is unsatisfactory, and therefore the results are not so significant. The 256 (313 — 57) who said "increases it" were right in one sense, and yet the very fact that they were satisfied with this answer indicates that they had not had much experience with this type of question.

Problem 13

For some reason, pupils were much more successful in answering this problem than they were in answering numbers 11 and 12. There is not so clear a choice between two possible answers as in number 11, and yet 70.6 per cent of those who answered it gave

the correct answer. In this case, we are justified in concluding that the answer was the result of something more than pure guessing. From the number omitted, we infer that the pupils considered number 12 the easiest of the three, and that they considered numbers 11 and 13 about equal in difficulty.

Problem 14

The omission of the zero value of I in this problem was not scored as an error, since it was felt that the pupil might omit the zero value because of a knowledge of the physics problem back of the formula. It is probable that the majority of those who omitted it did so carelessly or because they did not realize that zero is a root of the equation; but they were given the benefit of the doubt. There were 37 who gave the zero value only for the answer, while 2 gave zero for one answer and then gave a wrong answer for the other value. There were 12 out of the entire 859 who had both answers correct. If we score on this basis, this has by far the lowest per cent of correct answers of any problem in the test. Even with the liberal scoring indicated above, the problem is near the difficult end of the distribution. Out of the 360 who gave the wrong answer to this problem, 178 put I on the right-hand side of the equation in expressing the value of I.

Problem 15

More than half (51 per cent) of the pupils who missed this problem confused subscripts and exponents. It is admitted that many who did this knew the difference, and made the mistake through carelessness. Nevertheless, the frequency of this error can be attributed to the fact that the pupils had not had sufficient experience with subscripts to appreciate the danger of so confusing them. If more problems were given in which pupils were required to work with subscripts, they would learn to avoid confusing them with exponents.

Problem 16

If we do not consider the strict method of scoring number 12, number 16 is the most difficult problem in the test according to either method of computing the per cent of *corrects*. The most frequent error was in leaving p on the right-hand side of the equation. Of the 314 who gave the wrong answer, there were 169 who made this error. A similar error was noted in number

14. The fact that the unknown appears in two or more terms makes these two problems especially difficult. Solving for a fraction instead of solving for a single letter, being an unusual procedure, also makes the problem more difficult. In numbers 9, 10, 11, 12, and 13 it is possible that principles are involved in which some of the pupils had not been drilled. This is not true of number 16. It is simply an unusual and confusing combination of fundamental principles that are taught in all classes.

Problem 17

Considering the per cent of trials which were correct, it is evident that this problem appears among the easier ones of the test. It should be remembered, however, that it was not scored as wrong when m' or x was left in both numerator and denominator.

Problem 18

This problem, according to the tabulation of per cents correct, is but slightly more difficult than number 17. This is another case where the pupil is asked to solve for a ratio. This seems to have been the cause of a considerable part of the difficulty. Of the 195 who gave the wrong answer, there were 54 who had the answer inverted. It is not likely that all the inversions are due to having to solve for a ratio, but this peculiarity of the problem doubtless increased the number of times this error occurred. There were 19 who left either W or F on the right-hand side of the equation. The appearance of both r and R in this formula seems to have caused considerable confusion. There were 18 pupils who wrote $2r$ in the numerator where they should have written $2R$. Many formulas have R and r, or I and i, or M and m, or some such combination occurring in one formula. Suggestions for preparing pupils for this situation were made in Chapter III.

In most schools, this test was given soon after the beginning of the second semester to pupils who had just completed their third term of algebra. This means that the majority of them had not studied physics, but we have no assurance that none of them had studied physics. The training of the few who may have had one or two semesters of physics may have raised the average slightly, but practically all of the correct solutions must be due to training in high-school mathematics.

A summary of the errors in these problems shows that the cause

of the most errors in solving formulas is the failure to realize that in determining the value of a certain letter, one must not use that letter in the expression which represents its value. In numerical problems, the misplacing of the decimal point is responsible for more wrong answers than any other one error. The correct answer inverted is next in order of frequency. Both of these mistakes could have been avoided in the numerical problems if the pupils had appreciated the importance of roughly checking answers, since both errors gave far from the correct results.

CHAPTER VII

SUMMARY AND CONCLUSIONS

1. Previous investigations concerning the relationship between high-school mathematics and other subjects have dealt almost exclusively with facts and skills.

2. In general, there is close agreement between the findings in this investigation and the tabulated results from the questionnaires discussed in Chapter I which were sent out to teachers of college subjects by the National Committee, by E. L. Thorndike, and by Koch and Schlauch.

3. The frequency of certain words in the mathematical vocabulary determined by a careful word count of Stewart's textbook suggests the advisability of additional emphasis in high-school mathematics classes upon certain concepts. Among the words and expressions which suggest important concepts in physics that might profitably receive more emphasis in mathematics are the following: *axis, rotation, clockwise* and *counter-clockwise motion, positive* and *negative, constants* and *variables, maximum* and *minimum, rates* and *speeds,* and the series of words connected with variation and proportionality (direct and inverse).

4. College subjects dealing with quantitative data make use of an extensive symbolic language quite different from that used in high-school mathematics. The practice so generally followed in high-school mathematics of representing points by capital letters and quantities by small letters is not observed. Greek letters or initial letters (either capital or small) with primes or subscripts are used in preference to the customary x, y, z, a, b, and c of algebra. Such formulas as $\frac{W}{F} = \frac{2R}{R-r}$ and $\frac{I}{i} = \frac{R}{r}$, in which a letter appears both as a capital and a small letter are frequent in physics. Since Thorndike concludes from his investigation reported in Chapter III, page 25, that "Any disturbance whatsoever in the concrete particulars reasoned about will interfere with reasoning," and since the records of responses on numbers 15 and

18 of the test show clearly that pupils were handicapped by the use of subscripts and by the use of r and R in the same formula, it is recommended that high-school pupils be given practice with Greek letters, capital letters, primes, and subscripts in situations similar to those listed in this dissertation.

5. Many special symbols such as v for average velocity, Ca^{++} for calcium with two positive charges to an ion, and $\sigma_{1 \cdot 23}$ for standard deviation in trait 1 when traits 2 and 3 are held constant, are used in physics, chemistry, and statistics. It is impractical and inadvisable to attempt to teach these symbols in high-school, but it is very desirable that the mathematics courses be so presented in high school that pupils will fully appreciate the possibilities of a symbolic language. The teacher may even emphasize the fact that symbols are merely conventions by improvising some symbols to be used in the classroom as illustrations of the advantages of such a shorthand.

6. Acceptable authorities and the practice in the textbooks examined indicate that it is advisable (*a*) to discontinue the use of the double colon in a proportion, (*b*) to use letters rather than numbers to designate angles in a geometric figure, (*c*) to use the sign $\propto$ for *varies as.*

7. The arithmetic computations found in college subjects dealing with quantitative data are more complicated than the types of computation now generally advocated for courses in arithmetic. Very large numbers, very small numbers, decimal fractions, and common fractions enter into formulas in such a manner that the results are usually fractional rather than integral. The equation $(0.000012)(100)t = (0.000018)lt$, in which the student is required to solve for l, and the equation $2 = \frac{4(9.86965)5/32}{T^2} \cdot \frac{1}{12}$ to be solved for T are excellent illustrations.

8. There are comparatively few geometric facts involved in the subjects herein investigated, and these are the more simple and fundamental facts upon which ample drill is usually given in high school.

9. The ability to make scale drawings including the use of the protractor in making angles of a given size and in determining the size of given angles is essential.

10. Diagrammatic representation of relationships by means of *line graphs* and otherwise is a prominent feature of mathematics

which is used in physics and other college subjects. The ability to *read off* values from a graph is very essential in all subjects dealing with quantitative data, and should be emphasized in high school.

11. The meaning of *sine, cosine,* and *tangent,* and the ability to use tables of these functions constitute the amount of trigonometry that is absolutely essential in physics. The other college subjects use even less. Without any attempt to depreciate the value of trigonometry as ordinarily taught, it must be admitted that *numerical trigonometry* as now taught in many junior high schools is adequate preparation for college freshman physics in so far as mere subject matter is concerned. The use of other tables, such as those for square roots, is also essential and might well be taught along with the tables of the natural functions.

12. Standard form numbers represent the one phase of the theory of exponents with which students of physics and chemistry must be familiar. Addition, subtraction, multiplication, and division of standard form numbers appear with sufficient frequency to warrant the teaching of these four operations in high school. According to the record for Problem 9 in the test of Chapter VI, page 80, standard form numbers are not being efficiently taught at present.

13. The algebraic equations in one variable occurring in physics are not difficult, but they are unusual in the following respects: (*a*) The variable itself is unusual, being a Greek letter or some initial letter with perhaps a prime or a subscript. (*b*) A combination of decimal fractions and common fractions of unusual forms frequently occurs in the same problem. (*c*) The variable occurs in an unusual position as illustrated in the following:

$$20 = 2.2 + 5(0.1 + R), \; 1.9 = 2\pi\sqrt{\frac{90}{g}}.$$

(*d*) The answer is unusual, frequently being so large or so small that it is expressed as a standard form number. In the majority of cases, it is a fraction varying in accuracy from one significant digit at the right of the decimal point to seven significant digits at the right of the decimal point.

14. There are no complicated polynomials, complicated parentheses, complicated fractional equations, or complicated radicals, found in the texts examined.

15. The general quadratic equation such as $3x^2 + 5x + 7 = 0$

does not occur in Stewart's text, but three such equations were found in other texts. They are not frequent, and when they do occur, they are not put in such a special form that they are factorable. The pure quadratic such as $5x^2 = 90$ occurs frequently.

16. The customary form of simultaneous equations does not occur, but the following unusual form appears:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{5}$$
$$\frac{p}{q} = \frac{1}{6}.$$

Reference to Problem 7 in the test of Chapter VI, page 80, is convincing that this type of problem is not well understood.

17. In the transformations of formulas, there are no complicated manipulations, but merely those that are unusual. The only factoring that occurs is that of taking out a common monomial factor, either fractional or integral. The solution of an equation for a ratio instead of a single letter is one of the unusual requirements. The record of test results on problems 16 and 18 in Chapter VI, page 80, indicates that pupils were not efficient in this type of problem.

18. The general processes and methods of procedure which are listed below are arranged in the order of frequency of use in the expository material and in the solution of problems in Stewart's text. Theoretically the first two items apply to every problem; the number in parentheses after the statement of the other items indicates the frequency of use. This order is not necessarily the order of importance.

(1) The critical reading of the problem in order to get the exact meaning.

(2) Roughly checking the answer to see whether it is reasonable.

(3) Determining the appropriate formula and making the numerical substitution in it. (428)

(4) Changing from one unit of measure to another by means of a known ratio. (239)

(5) Reduction of data to the proper units of measure before substitution in the appropriate formula. (154)

(6) Numerical substitution in the appropriate verbal law or definition. (104)
(7) Translation of a verbal law or definition into a mathematical formula. (85)
(8) Transferring from the type or illustrative problem to the problem as stated, and then planning an analogous solution. (64)
(9) Unitary analysis. (64)
(10) Mathematical statement of a law which has not been expressed in words. (63)
(11) The resolution of a problem into two or more component parts, each part being dependent upon the one preceding. (55)
(12) A certain amount of generalization from the type or illustrative problem in order to transfer to the new problem, and then the planning of an analogous solution. (53)
(13) Making use of a fact, technique, or procedure that has been suggested by a preceding problem. (27)
(14) The use of the index of the text to find laws and definitions previously stated. (27)
(15) Practical interpretation of a law or relationship symbolically stated. (24)
(16) Solution of a problem for the ratio between two quantities when the value of neither quantity is known. (19)
(17) Discarding irrelevant data. (14)
(18) Interpretation of a formula when certain values or limitations are assigned to a letter or letters of the formula. (14)
(19) Interpretation of a negative root of an equation that has been established to solve a practical problem. (13)
(20) Using a verbal rule as a stimulus to a process carried on in imagination. (7)
(21) The resolution of a problem into two or more independent parts which are finally combined for the required answer. (6)
(22) Interpretation of the meaning of a negative value of a letter which is substituted in a formula. (5)
(23) Generalization from a numerical problem to a literal formula. (3)

In conclusion, the following points should be carefully considered:

1. If a certain fact or skill has been found to be necessary, even though but once, the student should know the fact or skill in question, but it should not be inferred that the high school should teach it. If the fact or skill is known to be useful and beneficial for other reasons, then its being found in physics or chemistry is an additional reason for emphasis.

2. The statement that a certain fact or skill was seldom or never found in this investigation does not imply that it is unnecessary. It may be very essential in some other connection. It may be an integral part of *mathematics as a language* even though we do not find occasion for its immediate use. A statement that it was seldom if ever found, however, may suggest the possible advisability of a shift of emphasis. For example, no ordinary simultaneous equations and no factoring except the taking out of a common monomial factor were found, but the translating of a verbal law into a mathematical formula was found to be necessary eighty-five times. It would seem that a shift of emphasis in this case might be profitable, not in physics only but in general.

3. From all available data, it seems that so far as subject matter is concerned, increased emphasis should be given to the following:

(1) Accuracy in computation with especial reference to decimal points and signs.
(2) Manipulation of formulas using capital letters, Greek letters, primes, and subscripts.
(3) Simple fractional equations.
(4) Simple radical equations.
(5) Simple applications of parentheses.
(6) Simultaneous equations in fractional forms.
(7) Fundamental operations with standard form numbers.
(8) Solving for a ratio between two letters.
(9) Proportion and variation as two phases of the same subject.
(10) Scale drawings.

4. Even though a pupil has been thoroughly trained in all of the facts and skills of high-school mathematics, we have no assurance that he will be successful in the college subjects dealing with quantitative data. Increased emphasis upon many of the general

processes and methods of procedure listed in item 18 of this chapter is probably more essential than increased emphasis upon subject matter.

Under the most favorable conditions, it will be necessary for the college teachers to assist students in adjusting themselves to the new situations even though the transfer be merely from one mathematical situation to another. It does seem reasonable, however, to hope that from the findings of this study, there may result certain shifts of emphasis which will decrease the difficulties of college students who enter beginning courses dealing with quantitative data, and which will also be fully as valuable as they now are for students not contemplating college work.

APPENDIX

TABLES FROM OTHER INVESTIGATIONS

Table 1 is reproduced from the study made by Koch and Schlauch reported in Contributions to Education, Volume 1, page 353, of the New York Society for the Experimental Study of Education.

TABLE 1

TOPICS LISTED IN ORDER OF IMPORTANCE

Group 1

Linear equations in 1 unknown	Measurement of areas
Simple formulas	Geometric reasoning
Ratio and proportion	Negative numbers

Group 2

Quadratic equations in 1 unknown	Checking results
Measurement of volumes	Graphic solution of problems
Simultaneous linear in 2 unknowns	Theory of exponents

Group 3

Literal equations	Simultaneous linear, more than 2 unknowns
Significant figures	Numerical trigonometry
Similarity of figures	Simultaneous 1 linear, 1 quadratic
Computation with approximate data	Variation
Use of logarithms in computation	Graphic representation, statistics

Group 4

Congruence, equality, inequality	Arithmetical progression
Binomial theorem	Geometric progression

Group 5

Statistics, elementary concepts	Simultaneous quadratics
Theory of logarithms	Short cuts in computation
$y = mx + b$	Statistics, frequency distribution
Plane trigonometry	$y = ax^2 + bx + c$
Use of slide rule	Imaginary numbers

TABLE 1 (*Continued*)

Group 6

Use of other tables than logarithms	Combinations and permutations
Analytic geometry of straight line	Analytic geometry of circle
Theory of probability	Polar coördinates
Mathematics of annuities	Statistics, correlation
Equations higher than second degree	Conic sections

The following table is reproduced from page 46 of the National Committee Report. The order was determined by taking the higher of the two ratings by the teachers of the physical sciences and the teachers of the social sciences in each case. The column headed E indicates the per cent who considered the topic *essential*. The column headed E+ C indicates the per cent who considered the topic either *essential* or *of considerable value*.

TABLE 2

TOPICS IN ORDER OF VALUE AS PREPARATION FOR ELEMENTARY COLLEGE COURSES

	E	E+C
Linear equations in one unknown	98	100
Simple formulas—their meaning and use	93	98
Ratio and proportion	84	92
Negative numbers—their meaning and use	79	84
Quadratic equations in one unknown	78	93
The linear function: $y = mx + b$	78	92
Simultaneous linear equations in two unknowns	71	95
Numerical trigonometry—the use of the sine, cosine, and tangent in the solution of simple problems involving right triangles	68	89
Demonstrative geometry	68	83
Use of logarithms in computation	62	91
Graphs as a method of representing dependence	62	78
Computation with approximate data—rational use of significant figures	61	97
The quadratic function: $y = ax^2 + bx + c$	59	80
Plane trigonometry—usual course	57	84
Graphic representation of statistical data	57	82
Statistics—meaning and use of elementary concepts	55	91
Variation	50	63
Statistics—frequency distributions and curves	47	80
Graphic solution of problems	45	65
Literal equations	43	61
Simultaneous linear equations in more than 2 unknowns	43	72
Simultaneous equations, one quadratic, one linear	40	64
Theory of exponents	36	67
Binomial theorem	35	67
Analytic geometry of the straight line	34	71
Theory of logarithms	34	60
Statistics—correlation	33	80
Analytic geometry—fundamental conceptions	32	77
Simultaneous quadratic equations	31	50
Analytic geometry of the circle	29	72
Short-cut methods of computation	29	65
Use of tables in computation (other than logarithms)	24	69
Use of slide rule	24	63
Arithmetic progression	23	52
Geometric progression	23	48
Imaginary numbers	23	44
Probability	20	55
Conic sections	18	59
Polar coördinates	18	44
Empirical curves and fitting curves to observations	12	50
Equations of higher degree than the second	10	42

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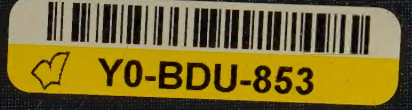
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